MATH 116-028 QUIZ 10 / 8 Apr 2008 Name:______________________________

1. Recall that the Taylor series for $\sin(x)$ is $\sum \frac{(-1)^n x^{2n+1}}{(2n+1)!}$. Find the Taylor series for $\text{Si}(x) = \int_0^x \frac{\sin(t)}{t}\,dt$. (3 points)

Solution: We know that $\sin(x) = \sum \frac{(-1)^n x^{2n+1}}{(2n+1)!}$, so $\frac{\sin(t)}{t} = \sum \frac{(-1)^n t^{2n}}{(2n+1)!}$. Integrating, we have

$$\text{Si}(x) = \int_0^x \frac{\sin(t)}{t}\,dt = \int_0^x \sum \frac{(-1)^n t^{2n}}{(2n+1)!}\,dt.$$

We assume that we can integrate term-by-term, to get

$$\text{Si}(x) = \sum \int_0^x \frac{(-1)^n t^{2n}}{(2n+1)!}\,dx = \sum \frac{(-1)^n x^{2n+1}}{(2n+1)(2n+1)!}.$$

We could also work this out with an expanded form of the series: $\text{Si}(x) = \int_0^x (1 - \frac{t^2}{3!} + \frac{t^4}{5!} - \cdots + \frac{(-1)^n t^{2n}}{(2n+1)!} + \cdots)\,dt = x - \frac{x^3}{3\cdot 3!} + \frac{x^5}{5\cdot 5!} - \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)\cdot(2n+1)!} + \cdots$. Note that because $\text{Si}(x)$ is defined as the integral from $t=0$ to $t=x$ we don't have a constant of integration in this problem.

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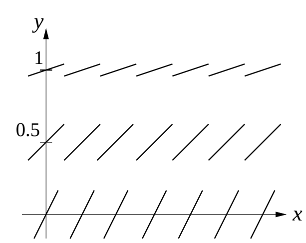
2. Suppose that we know that $\frac{dy}{dx} = f(y)$ for some function $f(y)$. Also suppose that we approximate the solution to this differential equation, with initial condition $y(0)=0$, with Euler's method and $\Delta x = 0.5$. If we find $y(0.5) \approx 1$, $y(1) \approx 1.5$, $y(1.5) \approx 1.75$, and $y(2) \approx 1.875$, (4 points)

a. What is $\frac{dy}{dx}$ at $y=0$, $y=0.5$, and $y=1$?

b. Give a rough sketch of the slope field of this differential equation.

Solution: We know that in general Euler's method gives $y(x+\Delta x) = y(x) + \Delta x f(y(x))$ (that is, $y_{n+1} = y_n + \Delta x f(y_n)$). Thus we know that $y(0.5) = 1 = 0 + 0.5 \cdot f(0)$, and thus $f(0) = \frac{dy}{dx}\big|_{x=0} = 2$. Similarly, with $y(1) = 1.5$, we have $y(1) = 1.5 = 1 + 0.5 \cdot f(0.5)$, so $f(0.5) = \frac{dy}{dx}\big|_{x=0.5} = 1$. And finally, we have $y(1.5) = 1.75$, so $y(1.5) = 1.75 = 1.5 + 0.5 \cdot f(1)$, and $f(1) = \frac{dy}{dx}\big|_{x=1} = 0.5$. Then we know that the slope of solutions to the differential equation are the same at any given y value, so that we have the slope field shown to the right.



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3. Find all solutions to the differential equation $\frac{1}{t}\frac{dp}{dt} + p = 2$. (3 points)

Solution: Rearranging the equation, we have $\frac{dp}{dt} = t(2-p)$. Note that $p=2$ is therefore a solution. If $p \neq 2$, then $\frac{dp}{2-p} = t\,dt$, so that $-\ln|2-p| = \frac{1}{2}t^2 + C$. Solving for p, we have $p = 2 - Ae^{-t^2/2}$. Thus solutions are $p=2$ or $p = 2 - Ae^{-t^2/2}$.