Some Remaining Review Problems—Solutions

(This is neither comprehensive nor guaranteed to be useful.)

1. Find the center of mass $\overline{x}$, $\overline{y}$ of the solid formed by the region bounded by $x = 0$, $x = 2$, $y = 1 + e^{-x}$ and $y = 1 - e^{-x}$,

a. Rotated about the y-axis, if its density is $\delta(y) = y$ (mass units/unit volume).

Solution—(Draw a figure to see the shape.) Rotated about the y-axis, with a density $\delta(y)$ that only depends on y, we can see by symmetry that $\overline{x}$ must be zero. To find $\overline{y}$ we need to find the mass first. A volume slice at a height y with a thickness Δy is $V_{sl} = \pi(\ln(1-y))^2\,\Delta y$ if $0 \le y \le 1 - e^{-2}$, $V_{sl} = \pi(2^2)\Delta y$ if $1 - e^{-2} \le y \le 1 + e^{-2}$, and $V_{sl} = \pi(\ln(y-1))^2\,\Delta y$ if $1 + e^{-2} \le y \le 2$. So the mass is

$$\begin{aligned}\int_0^2 \delta(y)\,V_{sl} &= \int_0^{1-e^{-2}} y\cdot\pi(\ln(1-y))^2\,dy + \int_{1-e^{-2}}^{1+e^{-2}} y\cdot\pi(2)^2\,dy + \int_{1+e^{-2}}^{2} y\cdot\pi(\ln(y-1))^2\,dy \\ &= 4\pi(1-3e^{-2}).\end{aligned}$$

Then $\overline{y}$ is the moment over the mass, where the moment is found by repeating the mass calculation with an additional factor of y:

$$\begin{aligned}\text{moment} &= \int_0^2 y\cdot\delta(y)\,V_{sl} \\ &= \int_0^{1-e^{-2}} y^2\cdot\pi(\ln(1-y))^2\,dy + \int_{1-e^{-2}}^{1+e^{-2}} y^2\cdot\pi(2)^2\,dy + \int_{1+e^{-2}}^{2} y^2\cdot\pi(\ln(y-1))^2\,dy \\ &= \frac{112}{27}\pi - \frac{28}{27}\pi e^{-6} - 12\pi e^{-2}.\end{aligned}$$

Taking the moment over the mass, we get $\overline{y} \approx 1.06127$.

b. Rotated about the x-axis, if its density is $\delta(x) = x$ (mass units/unit volume).

Solution—(Similarly, draw a figure to see the shape.) This is easier, because we don't have to break the region up to do the integral. In this case we get

$$\begin{aligned}\text{mass} &= \int_0^2 x\,\pi\left((1+e^{-x})^2 - (1-e^{-x})^2\right)dx = 4\pi(1-3e^{-2}), \\ \text{moment} &= \int_0^2 x^2\,\pi\left((1+e^{-x})^2 - (1-e^{-x})^2\right)dx = 8\pi(1-5e^{-2}).\end{aligned}$$

and so $\overline{x} \approx 1.08864$.

2. If, for the region in **(1a)**, we had $\delta(x) = 3 + x$, can we find $\overline{x}$? $\overline{y}$? Why or why not?

Solution—We can't calculate either of these easily with the tools that we have at our disposal in Math 116. You can see this by thinking about how we have to slice the region. The fact that density is a function of x means that we want to slice the region so that slices are perpendicular to the x-axis. This is fine, but the solid doesn't have any nice shape in that direction, so it becomes very difficult to figure out the mass in this case. Of course, if the density only depends on x and the object is symmetric with respect to y ($y = 1$ in this case), then we can deduce $\overline{y}$ from a symmetry argument: $\overline{y} = 1$. But if we were to try to find the moment needed to calculate $\overline{y}$, it will require us to have an integral with a factor of y in it, which would mean that we have to slice the region perpendicular to the y-axis—which doesn't work at all, because we already know we have to slice it with respect to x because of the density. Thus for $\overline{x}$ we're stuck because it's very difficult to figure out what the slice of volume looks like to find the mass, and if we try to find $\overline{y}$ using calculus the problem is further complicated because we can't logically set up an integral for the moment that we need to find the center of mass.

3. Let $p(x)$ be a pdf, with $a \le x \le b$. Let $Q(x)$ be an antiderivative of $p(x)$. Show that the cdf of $p(x)$ is given by $P(x) = Q(x) - Q(b) + 1$.

Solution—We know that $P(x) = \int_a^x p(t)\,dt = Q(x) - Q(a)$. Then $\int_a^b p(t)\,dt = Q(b) - Q(a) = 1$, so $Q(a) = Q(b) - 1$. Thus $P(x) = Q(x) - Q(b) + 1$.

4. Let $p(x)$ be a pdf for the distribution of GPAs, x, earned by University of Michigan squirrels.

a. What is the domain of $p(x)$?

Solution—The domain is all reasonable GPAs, so $0 \le x \le 4$.

b. Sketch a reasonable graph that could be $p(x)$.

Solution—Any graph will do. It must be defined for $0 \le x \le 4$, must have $p(x) \ge 0$, and the area under the graph must be one.

c. What is the meaning of the statement $p(2) = 0.05$?

Solution—This is tricky. We only have meaning associated with the area under the pdf, not with a specific value of the pdf. So to make a statement about meaning we have to relate the value to some area. Let's think about a small region on either side of $x = 2$: say, from $x = 1.9$ to $x = 2.1$. Then if $p(x)$ were close to constant on this region, we'd have that the area under the curve there would be $(0.2)(0.05) = 0.01$. Thus we can say that the fraction of squirrels having a GPA between 1.9 and 2.1 is approximately 0.01 (that is, about 1% of the squirrels).

d. If $P(x)$ is the cdf for this distribution, what is the meaning of the statement $P(2) = 0.953$?

Solution—This is easier: 95.3% of squirrels have a GPA of two or less.

5. Carefully determine if each of the following series converges or diverges.

a. $\sum \frac{x^n}{5^n + n^2}$, if $|x| \le 4$.

Solution—Use the ratio test; it converges.

b. $\sum_{n=2}^{\infty} \frac{1}{n \cdot \ln(n)}$

Solution—Use the integral test; it diverges, albeit slowly.

c. $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \cdot \ln(n)}$

Solution—Use the alternating series test; it converges.

6. Carefully explain why, if $\sum |a_n|$ converges, we are able to conclude that $\sum a_n$ also converges.

Solution—This is shown in the book, #67 on p. 463. Use the comparison test.

a. Carefully explain why, if $\sum a_n$ converges, we are unable to conclude that $\sum |a_n|$ also converges.

Solution—We can show this by example: $\sum \frac{(-1)^n}{n}$ converges while in absolute value it doesn't, while $\sum \frac{(-1)^n}{n^2}$ converges absolutely. Thus there are series that converge conditionally and others that converge absolutely, and we aren't able to infer convergence of $\sum |a_n|$ from the convergence of $\sum a_n$.