VECTOR CALCULUS FORMULAS TO KNOW AND LOVE

(from Chapter 17 in Stewart)

First, in all of the following:

- The notation $\mathbf{r}(t) = \overrightarrow{r}(t)$ indicates a *position vector* that specifies a **curve** C. We assume that $a \le t \le b$. For example, $\mathbf{r} = 3\cos(t)\,\mathbf{i} + 3\sin(t)\,\mathbf{j} + 0\,\mathbf{k}$, for $0 \le t \le 2\pi$.
- Then, $ds = |\mathbf{r}'(t)|\,dt =$ the arclength element along a curve, and $d\mathbf{r} = \mathbf{r}'(t)\,dt =$ the vector element along a curve.
- The notation $\mathbf{r}(u,v) = \overrightarrow{r}(u,v)$ indicates a *position vector* that is a function of *any two variables* and which specifies a **surface** S. We assume that the variables $(u,v) \in D$ for some domain D. For example, $\mathbf{r} = <r\cos(\theta), r\sin(\theta), 4 - r^2>$, for $0 \le r \le 1$, $0 \le \theta \le 2\pi$.
- Then $dS = |\mathbf{r}_u \times \mathbf{r}_v|\,du\,dv =$ the surface area element on a surface, and $d\mathbf{S} = \mathbf{r}_u \times \mathbf{r}_v\,du\,dv =$ the oriented vector surface area element (with vector normal to the surface).
- The function $f(x,y)$ or $f(x,y,z)$ is a *scalar function* that returns a value: e.g., $f(x,y,z) = 3xy + \sin(z)$.
- The vector function $\mathbf{F}(x,y)$ or $\mathbf{F}(x,y,z)$ is a *vector field* that returns a vector: e.g., $\mathbf{F} = 2xy\,\mathbf{i} + x^2\,\mathbf{j}$. We will also use the notation $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$ when referring to two-dimensional vector fields. For example, in the preceding vector field, $P = 2xy$ and $Q = x^2$.
- Because we write $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$ for two-dimensional $\mathbf{F}$, the following integral notations are equivalent: $\int \mathbf{F} \cdot d\mathbf{r} = \int P\,dx + Q\,dy$.

Make sure that the notations indicated above are obvious to you—to the point of your making the assumption that if you see $\mathbf{r}(t)$ *written somewhere you immediately think "this defines a curve C" and start thinking about what curve it is, etc.*

Theorems of Vector Calculus

- **The Fundamental Theorem of Line Integrals**

$$\int_C \overrightarrow{\nabla} f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

Note that this applies for *conservative* $\mathbf{F}$: $\int_C \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$.

- **Green's Theorem**

$$\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA = \oint_C \mathbf{F} \cdot d\mathbf{r}$$

Here D *is a region in the xy-plane*, and C *is the curve that bounds* D, oriented counterclockwise. Therefore: Green's theorem is **only applicable** to **closed** curves C that are in the xy-plane. Also note that for two-dimensional functions $\mathbf{F}$, $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \operatorname{curl}\mathbf{F} \cdot \mathbf{k}$—and that $\mathbf{k}$ is the normal vector for a region (surface) in the xy-plane.

- **Stokes' Theorem**

$$\iint_S \operatorname{curl}\mathbf{F} \cdot d\mathbf{S} = \oint_C \mathbf{F} \cdot d\mathbf{r}$$

Here $\mathbf{S}$ *is any bounded surface* S, and C *is the curve that bounds* S, oriented "counterclockwise." Therefore: Stokes' theorem is only applicable to the (odd) **cases when we want to find** $\iint_S \operatorname{curl}\mathbf{F} \cdot d\mathbf{S}$, and/or **cases when we want to find** $\oint_C \mathbf{F} \cdot d\mathbf{r}$ *and there is a "nice" surface S bounded by C.*

- **the Divergence Theorem (Gauss' Theorem)**

$$\iiint_E \operatorname{div}\mathbf{F}\,dV = \iint_S \mathbf{F} \cdot d\mathbf{S}$$

Here $\mathbf{S}$ is the **surface that bounds the volume** E. Therefore: the divergence theorem is most applicable to **cases when we want to find the flux through a closed surface**.