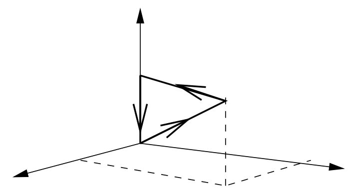
A VECTOR INTEGRAL PROBLEM EXAMPLE

Problem: Find $\int_C \mathbf{F}\cdot d\mathbf{r}$ around C if $F = <3xy, 2yz, z>$ and C is the piecewise linear curve that extends from $(0,0,0)$ to $(1,2,1)$ to $(0,0,1)$ to $(0,0,0)$.



Answer 1: The curve is sketched to the right. We notice that it is a *closed curve with a nice contained surface* S, namely the triangle inside the curve. Therefore, we can find the line integral by using *Stokes' theorem*: $\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S}$.

So let's use Stokes' theorem. To do this, we need to parameterize the surface to find $d\mathbf{S}$, we need to find $\operatorname{curl}\mathbf{F}$, and we need to integrate $\operatorname{curl}\mathbf{F}\cdot d\mathbf{S}$ over the surface. We'll do this in parts.

First, we know that $d\mathbf{S} = \mathbf{r}_u \times \mathbf{r}_v\, du\, dv$, so we need to find $\mathbf{r}$, a parameterization of the surface S. Noticing that the surface is just a cut-out from the plane $y = 2x$, we might choose to choose $\mathbf{r}$ to be the plane and then restrict the domain that we're looking at in that plane to the triangular piece. The plane can have any z value, so we can write it as $\mathbf{r}(x,z) = <x, 2x, z>$. What's the domain? We have to describe it in terms of x and z. Clearly, $0 \le x \le 1$, and z goes from the diagonal line to $z = 1$. The diagonal line has $z = 0$ when $x = 0$ and $z = 1$ when $x = 1$, so it must be $z = x$. Thus our finished parameterization is $\mathbf{r}(x,z) = x\,\mathbf{i} + 2x\,\mathbf{j} + z\,\mathbf{k}$ with $0 \le x \le 1$ and $x \le z \le 1$. Then $\mathbf{r}_u = \mathbf{r}_x = <1,2,0>$ and $\mathbf{r}_v = \mathbf{r}_z = <0,0,1>$, so $d\mathbf{S} = (\mathbf{r}_u \times \mathbf{r}_v)\, du\, dv = (\mathbf{r}_x \times \mathbf{r}_z)\, dx\, dz = <1,2,0> \times <0,0,1>\ dx\, dz = <2,-1,0>\ dx\, dz$.[1]

Second, we need to know $\operatorname{curl}\mathbf{F}$. This is just $\overrightarrow{\nabla} \times \mathbf{F} = -2y\,\mathbf{i} + 0\,\mathbf{j} - 3x\,\mathbf{k}$. Thus $\operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = -4y\, dx\, dz$. This needs to be evaluated on the surface, where $y = 2x$, so $\operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = -8x\, dx\, dz$.

Finally, we can evaluate the integral easily, finding $\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \int_0^1 \int_x^1 -8x\, dz\, dx = \int_0^1 8(x^2 - x)\, dx = -\frac{4}{3}$.

Answer 2: Of course, we could also simply evaluate the line integral. To do this, we'll break the curve C into three pieces: the curve C_1 going from $(0,0,0)$ to $(1,2,1)$, the curve C_2 going from $(1,2,1)$ to $(0,0,1)$, and the curve C_3 going from $(0,0,1)$ to $(0,0,0)$. For each of these, we find $\int_{C_i} \mathbf{F}\cdot d\mathbf{r}$ by first finding a parameterization $\mathbf{r}(t)$ of the curve, then finding $d\mathbf{r} = \mathbf{r}'(t)\, dt$, then taking the dot product and evaluating the integral.

On C_1, we can parameterize $\mathbf{r}(t) = < t, 2t, t >$, with $0 \le t \le 1$. Then $\mathbf{r}'(t) = < 1, 2, 1 >\ dt$, and $\mathbf{F}\cdot d\mathbf{r} = (3xy + 4yz + z)\, dt$. Of course, we're only interested in values of x, y and z on the curve C_1, so we can plug in $x = t$, $y = 2t$ and $z = t$ to get $\mathbf{F}\cdot d\mathbf{r} = (14t^2 + t)\, dt$. (Check that you get the same thing.) Then $\int_{C_1} \mathbf{F}\cdot d\mathbf{r} = \int_0^1 (14t^2 + t)\, dt = \frac{31}{6}$.

On C_2, we can parameterize $\mathbf{r}(t) = <1-t, 2(1-t), 1>$, with $0 \le t \le 1$. Then $\mathbf{r}'(t) = <-1,-2,0>\ dt$, and $\mathbf{F}\cdot d\mathbf{r} = (-3xy - 4yz)\, dt$. Again, we're only interested in values of x, y and z on the curve C_2, so we can plug in $x = 1-t$, $y = 2(1-t)$ and $z = 1$ to get $\mathbf{F}\cdot d\mathbf{r} = (-6(1-t)^2 - 8(1-t))\, dt$. Then $\int_{C_2} \mathbf{F}\cdot d\mathbf{r} = \int_0^1 (-6(1-t)^2 - 8(1-t))\, dt = -6$.

On C_3, we might take $\mathbf{r}(t) = <0,0,1-t>$, with $0 \le t \le 1$. Then $\mathbf{r}'(t) = <0,0,-1>\ dt$, and $\mathbf{F}\cdot d\mathbf{r} = -z\, dt$. Here $z = 1-t$, so $\mathbf{F}\cdot d\mathbf{r} = -(1-t)\, dt$. Then $\int_{C_3} \mathbf{F}\cdot d\mathbf{r} = \int_0^1 (t-1)\, dt = -\frac{1}{2}$.

The integral is then $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{C_1} \mathbf{F}\cdot d\mathbf{r} + \int_{C_2} \mathbf{F}\cdot d\mathbf{r} + \int_{C_3} \mathbf{F}\cdot d\mathbf{r} = \frac{31}{6} - 6 - \frac{1}{2} = \frac{31}{6} - \frac{36}{6} - \frac{3}{6} = -\frac{8}{6} = -\frac{4}{3}$.

Which of these is easier? In general, it's hard to say. There are some cases (in particular, *Green's theorem* and the *divergence theorem*) where one of the possible integrals is often easier to do (in Green's theorem, it's often more desirable to integrate $\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dx\, dy$ than $\oint_C \mathbf{F}\cdot d\mathbf{r}$, and in the divergence theorem it is almost always better to evaluate $\iiint_V \operatorname{div}\mathbf{F}\, dV$ than $\iint_S \mathbf{F}\cdot d\mathbf{S}$). In this case, it's not clear. A good rule of thumb is to try whichever formulation isn't given to you, or to start both and see which looks like it is going to be easier.

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[1] Note that the surface could also be characterized with $\theta = \arctan(2)$, in which case by using cylindrical coordinates we can get the alternate parameterization $\mathbf{r} = < r\cos(\theta), r\sin(\theta), z > = < \frac{1}{\sqrt{5}} r, \frac{2}{\sqrt{5}} r, z >$ with $0 \le r \le \sqrt{5}$ and $\frac{1}{\sqrt{5}} r \le z \le 1$. Check that you can see how this is derived and use it to obtain the same answer as above.