For all problems, *SHOW ALL OF YOUR WORK*. Partial solutions and problems with missing steps will be marked wrong. Continue your work on the back of the page or extra sheet at the end of the exam if you need additional space. *You do not need but may use the normal graphing calculator functions of any graphing calculator, but not any differential equations functionality it may have.*

1. Solve each of the following, explicitly if possible and implicitly otherwise. (33 points)

   a. $5t\frac{dz}{dt} = -10z + \frac{1}{t}\cos(4t)$, $z(\pi) = 2$.
   Solution: This is first order and linear, and it isn't separable, so we choose to use the method of integrating factors. Rewritten, the equation is

$$\frac{dz}{dt} + \frac{2}{t}z = \frac{1}{5t^2}\cos(4t),$$

   so the integrating factor is $e^{\int 2/t dt} = e^{2\ln(t)} = t^2$. Multiplying through by this, we get $\frac{d}{dt}(t^2z) = \frac{1}{5}\cos(4t)$. Integrating both sides, $t^2z = \frac{1}{20}\sin(4t) + C$, so $z = \frac{1}{20t^2}\sin(4t) + \frac{C}{t^2}$. Applying the initial condition $z(\pi) = 2$ gives $C = 2\pi^2$, so that our solution is

$$z = \frac{1}{20t^2}\sin(4t) + \frac{2\pi^2}{t^2}.$$

   b. $5z\frac{dz}{dt} = (z^2+1)^{1/2}\cos(4t)$.
   Solution: This problem is first order but non-linear, so we can't use integrating factors. Fortunately, it is separable, so we separate variables to get

$$\frac{5z}{(z^2+1)^{1/2}}dz = \cos(4t)dt.$$

   Integrating both sides, $5(z^2+1)^{1/2} = \frac{1}{4}\sin(4t) + C$, so that

$$z = \pm\sqrt{(\frac{1}{20}\sin(4t) + C)^2 - 1}.$$

   c. $y''' + 4y' = 0$, with $y(0) = 0$, $y'(0) = 1$ and $y''(0) = 0$.
   Solution: This is a third order equation, so essentially our only method is to guess $y = e^{rt}$. Plugging this in, we get $r^3 + 4r = r(r^2+4) = 0$. Thus $r = 0$ or $r = \pm 2i$, so the general solution is

$$y = C_1 + C_2\cos(2t) + C_3\sin(2t).$$

   The initial conditions require that $C_1 + C_2 = 0$, $2C_3 = 1$, and $-4C_2 = 0$, so $C_1 = C_2 = 0$ and $C_3 = \frac{1}{2}$. Our solution is then

$$y = \frac{1}{2}\sin(2t).$$

2. Consider the differential equation $4t^2y''(t) - ty'(t) + y(t) = 0$ (which we don't know how to solve). (15 points)

   a. Show that $y_1 = t^{1/4}$ and $y_2 = t$ are solutions to this differential equation (for $t > 0$).
   Solution: To show that these are solutions, we plug them into the equation. $y_1' = \frac{1}{4}t^{-3/4}$ and $y_1'' = -\frac{3}{16}t^{-7/4}$, so, plugging in, we have

$$\begin{aligned} 4t^2(-\frac{3}{16})t^{-7/4} - t(\frac{1}{4})t^{-3/4} + t^{1/4} &= -\frac{3}{4}t^{1/4} - \frac{1}{4}t^{1/4} + t^{1/4} \\ &= 0, \end{aligned}$$

   so it is a solution. Similarly, $y_2' = 1$ and $y_2'' = 0$, so plugging in gives $0 - t + t = 0$, and y_2 is also a solution.

   b. Write the general solution to the differential equation. What is true about y_1 and y_2 that allows you to do this? How do you know?
   Solution: The general solution to the differential equation is

$$y = C_1y_1 + C_2y_2 = C_1t^{1/4} + C_2t.$$

   We know we can do this because the problem is linear, and therefore a linear combination of any two linearly independent solutions gives the general solution. We need, of course, to be sure that y_1 and y_2 are linearly independent. This is easily done by noting that they aren't constant multiples (for no k is it the case that $t^{1/4} = kt$), or by finding that the Wronskian is nonzero (for $t > 0$, as indicated above):

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} t^{1/4} & t \\ \frac{1}{4}y^{-3/4} & 1 \end{vmatrix} = t^{1/4} - \frac{1}{4}t^{1/4} = \frac{3}{4}t^{1/4}.$$

   c. Find the particular solution for the differential equation if $y(1) = 3$ and $y'(1) = 2$.
   Solution: Plugging $t = 1$ in to the general solution, above, and its derivative, we get $C_1 + C_2 = 3$ and $\frac{1}{4}C_1 + C_2 = 2$. Subtracting the second from the first gives $C_1 = \frac{4}{3}$, so that $C_2 = \frac{5}{3}$. The particular solution is therefore

$$y = \frac{4}{3}\,t^{1/4} + \frac{5}{3}t.$$

3. Find $|z|$, $Arg(z)$ and the exponential (polar) form of $z = \left(\frac{1}{2-2i}\right)^8$. (6 points)
   Solution: Let $y = \frac{1}{2-2i} = \frac{1}{2}(\frac{1}{1-i})$. Multiplying the numerator and denominator of this by the complex conjugate of the denominator, $1+i$, gives $y = \frac{1}{2}(\frac{1+i}{2}) = \frac{1}{4}\,(1+i)$. This is a point in the first quadrant 45° ($\frac{\pi}{4}$ radians) from the x-axis and a distance $|y| = \frac{1}{4}\sqrt{1^2+1^2} = \frac{\sqrt{2}}{4}$ from the origin. Thus $y = \frac{\sqrt{2}}{4}\,e^{i\pi/4}$. Then

$$z = y^8 = (\frac{\sqrt{2}}{4})^8 e^{2i\pi} = \frac{2^4}{2^{16}} = \frac{1}{2^{12}} = \frac{1}{4096}.$$

   This is the exponential form of z; its modulus $|z| = \frac{1}{4096}$ and argument $Arg(z) = 0$ (or 2π).

4. An alert turtle named Yert observes that tin organ pipes decay with age as a result of a chemical reaction which is catalyzed by the decayed tin. As a result, the rate at which the tin decays is proportional to the product of the amount of tin left and the amount that has already decayed. Let M be the total amount of tin before any has decayed. (14 points)

   a. Write a differential equation for the amount of decayed tin, $p(t)$. Be sure it is clear why your equation has the form it does.

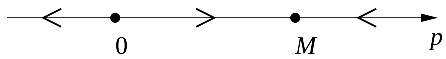
   Solution: The equation is

   $$p'(t) = k(M - p(t))p(t).$$

   This says that the rate of change in the decayed tin, $p'(t)$ (which is the same as the rate at which tin decays) is proportional to (k) the product of the amount of tin left ($M - p(t)$) and the amount that has already decayed ($p(t)$). Exactly as we want.

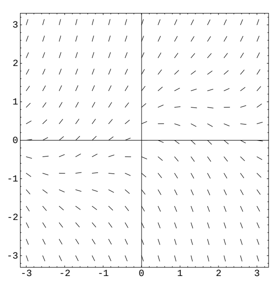
   b. Draw a phase diagram for your differential equation (take $M = 10$ and your constant of proportionality, $k = 2$, if you like) and explain what this tells you about the decay of the tin if initially none of it is decayed. How does this change if there is a very small amount decayed initially?

   Solution: This equation is autonomous, so it makes sense to talk about equlibrium points. The equilibrium points are where $p'(t) = 0$, which are when $p(t) = 0$ or $p(t) = M$. (M is an amount of tin, so $M > 0$. If $p(t) < 0$, $p'(t)$ is the product of a positive and a negative term, and so is negative. If $0 < p(t) < M$, $p'(t)$ is the product of two positive terms, and so is positive, and if $p(t) > M$, $p'(t) < 0$. Thus the amount of decayed tin $p(t)$ is decreasing, increasing and decreasing respectively in these regions (never mind that it doesn't make sense for $p(t)$ to be less than 0 or greater than M), and our phase diagram is the following:



   Because $p(t) = 0$ is an equilibrium solution, if there is no decay initially the tin will never decay. However, if there is a small amount of decay the phase diagram tells us that eventually all of the tin will decay.

5. Sketch solutions to the differential equation whose direction field is shown to the right through the initial conditions $y(0) = 1$ and $y(-2) = -1$. Then carefully explain how you know that the direction field is *not* the the direction field for the differential equation $\frac{dy}{dx} = x - \sin(y)$. (8 points)



   Solution: Note that for the given differential equation, at $(0, 1)$ the slope is $\frac{dy}{dx} = -\sin(1)$ and at $(1, 0)$ it is $\frac{dy}{dx} = 1$. Thus at these points the direction field lines should have negative and positive slope, respectively. However, this is clearly not the case for the given direction field. Thus the direction field can't be that for this differential equation. *(Sketching the solutions is left for the reader.)*

6. The initial value problem $yy' = y - y^2$, $y(0) = 0$ has two solutions. (14 points)
   a. Does this conform to or contradict the existence and uniqueness theorem for first-order ordinary differential equations? (Recall that the theorem starts "*For $y'(x) = f(x, y)$, if the function $f(x, y)$ is...*")
   Solution: This must conform to the theorem (or else it wouldn't be a theorem, after all). In this case, we have the differential equation

$$y' = \frac{y - y^2}{y} = f(x, y),$$

   for which the function $f(x, y)$ is discontinuous when $y = 0$. For the theorem to apply f must be continuous at and near the initial condition. Thus the theorem can't be applied, it therefore says nothing about this initial value problem, and there can be two solutions without perturbing us in the least.

   b. Find the two solutions to $yy' = y - y^2$, $y(0) = 0$.
   Solution: By inspection, one such solution is $y = 0$. If $y \neq 0$, we can divide it out to get $y' = 1 - y$. The solution to this is $y = Ce^{-x} + 1$, and to satisfy the initial condition $C = -1$. Thus a second solution is $y = 1 - e^{-x}$.

7. A not entirely defensible numerical method for approximating the solution to the differential equation $\frac{dy}{dx} = f(x, y)$ might use the iteration formula $y_{n+1} = y_n + hf(x_n, z)$, where $z = y_n + \frac{1}{2}hf(x_n, y_n)$. Approximate $y(0.2)$ using $h = 0.1$ and this numerical method (tentatively called "Yert's method") if $\frac{dy}{dx} = 2 - xy^2$ and $y(0) = 0.5$. (10 points)
   Solution: Let's make a table of values:

n	x_n	y_n	$f(x_n, y_n)$	z	$f(x_n, z)$	y_{n+1}
0	0	.5	$2 - 0 = 2$	$.5 + .05 \cdot 2 = .6$	$2 - 0 = 2$	$.5 + .1 \cdot 2 = .7$
1	.1	.7	$2 - .1 \cdot .7^2 = 1.951$	$.7 + .05 \cdot 1.951 = .798$	$2 - .1 \cdot .798^2 = 1.936$	$.7 + .1 \cdot 1.936 = .894$
2	.2	.894				

   So $y(0.2) \approx 0.894$.