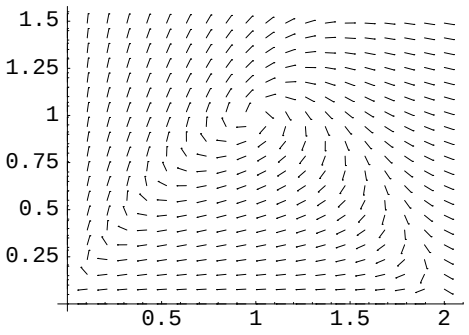
Math 216–050 Final, 13 December 2002 Name: _____

Prof. Gavin LaRose **page 1 of 4** Lab section: _____ (8,9am,12,1pm = 51-54)

Please do *not* fill in:	problem:	1	2	3	4	5	6
	total pts:	9	18	26	8	16	27
	score:						

For all problems, *SHOW ALL OF YOUR WORK*. While partial credit will be given, partial solutions that could be obtained directly from a calculator or a guess are worth no points. **Continue your work on the back of the page or extra sheet at the end of the exam if you need additional space**. *You do not need but may use the normal graphing calculator functions of any graphing calculator, but NOT any differential equations functionality it may have.*

1. The direction field for a system $\frac{dx}{dt} = F(x,y)$, $\frac{dy}{dt} = G(x,y)$ is shown to the right.

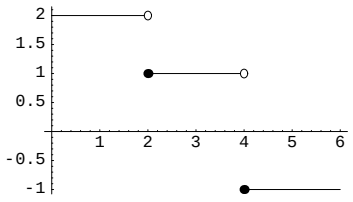


   a. Sketch the x- and y-nullclines on the graph. Label which are the x- and which the y-nullclines. (6 points)

   b. Mark all critical points on the graph. (3 points)

2. Find each of the following.

   a. *Use the definition* of the Laplace Transform to find $\mathcal{L}\{f(t)\}$ if $f(t)$ is given by the figure shown to the right, below.



(6 points)

   b. Write a formula for the function $f(t)$ shown in the figure, and use the transforms in the table at the end of the final to confirm your result from (a). (6 points)

   c. Find $\mathcal{L}^{-1}\{\frac{1}{e^{4s}(s^2+6s+13)}\}$, using the transforms in the table at the end of this final. (6 points)

3. Solve each of the following.

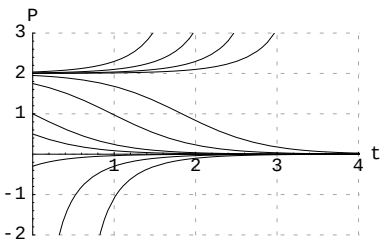
a. $xy' = -2y + 4x^2$, $y(1) = 5$. (5 points)

b. $yy' = 2x^2y^{-1} + y^{-1}$, $y(0) = 3$. (5 points)

c. $y'' + 4y' + 4y = 2x + 3e^{-x}$. (6 points)

d. $y'' + 3y' + 2y = \delta_3(t)$, $y(0) = 0$, $y'(0) = 4$. (6 points)

4. Several solution curves for an equation $\frac{dP}{dt} = f(P)$ are shown in the figure to the right.



   a. Sketch a phase diagram for this differential equation. Indicate all equilibrium points and their stability. (4 points)

   b. Write a differential equation that could give your phase diagram and the solution curves shown in the figure. (4 points)

5. A tortoise visiting from the Galápagos Islands has brought along two solutions to a differential equation. However, being a bit forgetful, it has misplaced the equation they solve. If the two solutions are $y_1 = e^{-t}\cos(2t)$ and $y_2 = e^{-t}\sin(2t)$, answer the following questions.

   a. Write a differential equation of lowest order which the solutions could solve. How do you know the solutions y_1 and y_2 solve the equation you give? (6 points)

   b. Could the solutions y_1 and y_2 be used to write a general solution to your differential equation? Why or why not? What would the general solution be? (6 points)

   c. The tortoise is now curious to know if the equation you wrote down is damped or undamped (and if damped, what type of damping it has), and whether it is forced or unforced. Is it possible to have a forcing that would produce beats, resonance or practical resonance? Why or why not? (4 points)

6. Suppose you are given a mass-spring system with a not-particularly-physically-reasonable spring, which is modeled by the equation $x'' + 2x' + 3(x - x^2) = 0$.
   a. Write this as a system of two first-order differential equations. (4 points)

   b. Find all equilibrium solutions to the problem. What are the mass' position and velocity at each of your equilibrium solutions? (6 points)

   c. Find the behavior near each of the equilibrium solutions you found. (12 points)

   d. Sketch on a phase plane what you expect the trajectories starting with the initial conditions $x(0) = 0, x'(0) = 1$ and $x(0) = 0, x'(0) = 5$ to look like. Comment on whether the behavior you see is what you would expect for a mass-spring system or not. (5 points)

(Possibly) Useful Transform Rules

$f(t)$	$F(s)$	$f(t)$	$F(s)$
1	$\frac{1}{s}$	$f(t)$	$F(s)$
t^n	$\frac{n!}{s^{n+1}}$	$\int_0^t f(\tau)\,d\tau$	$\frac{F(s)}{s}$
e^{at}	$\frac{1}{s-a}$ $(s > a)$	$e^{at}f(t)$	$F(s-a)$
$\cos(kt)$	$\frac{s}{s^2+k^2}$	$u(t-a)f(t-a)$	$e^{-as}F(s)$
$\sin(kt)$	$\frac{k}{s^2+k^2}$	$\int_0^t f(\tau)g(t-\tau)\,d\tau$	$F(s)\cdot G(s)$
$u(t-a)$	$\frac{e^{-as}}{s}$	$tf(t)$	$-F'(s)$
$\delta(t-a)$	e^{-as}	$\frac{f(t)}{t}$	$\int_s^\infty F(\sigma)\,d\sigma$
$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$		