For all problems, *SHOW ALL OF YOUR WORK*. While partial credit will be given, partial solutions that could be obtained directly from a calculator or a guess are worth no points. Continue your work on the back of the page or extra sheet at the end of the exam if you need additional space. *You do not need but may use the normal graphing calculator functions of any graphing calculator, but NOT any differential equations functionality it may have.* If you need to borrow a graphing calculator, ask me.

1. For each of the following, set up the appropriate form of the particular solution y_p but *do not* determine the values of the coefficients in your solution. Be sure it is clear why your expression for y_p has the form it does.

   a. $y'' + 3y' + 2y = 4e^{-2x} - \pi e^{2x}$. (8 points)

   Solution: Note that the complementary homogeneous solution to the problem is $y_c = c_1 e^{-x} + c_2 e^{-2x}$. Using the Method of Undetermined Coefficients, we would guess $y_p = Ae^{-2x} + Be^{2x}$. However, the first of these terms is in y_c, so we modify our guess to

$$y_p = Axe^{-2x} + Be^{2x}.$$

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   b. $y'' + 4y = x(4 + 2\sin(2x))$. (8 points)

   Solution: Here, $y_c = c_1\cos(2x) + c_2\sin(2x)$. To see the correct form for y_p it's easiest to expand out the right-hand side of the equation to $4x + 2x\sin(2x)$. We would guess $y_p = Ax + B + Cx\cos(2x) + D\cos(2x) + Ex\sin(2x) + F\sin(2x)$, to capture all of the derivatives of the forcing terms, but the cosine and sine terms are present in the homogeneous solution. We therefore modify this to

$$y_p = Ax + B + Cx^2\cos(2x) + Dx\cos(2x) + Ex^2\sin(2x) + Fx\sin(2x).$$

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2. Use the eigenvalue method to solve the initial value problem $\vec{x} = \begin{pmatrix} 3 & 8 \\ 2 & -3 \end{pmatrix}\vec{x}$, $\vec{x}(0) = \begin{pmatrix} 7 \\ 3 \end{pmatrix}$. (16 points)

   Solution: Let $\vec{x} = \vec{v}e^{\lambda t}$. Then $\begin{pmatrix} 3-\lambda & 8 \\ 2 & -3-\lambda \end{pmatrix}\vec{v} = \vec{0}$, so $\det(\begin{pmatrix} 3-\lambda & 8 \\ 2 & -3-\lambda \end{pmatrix}) = \lambda^2 - 25 = 0$. Thus $\lambda = \pm 5$. If $\lambda = -5$, the matrix equation for $\vec{v}$ degenerates to $v_1 + v_2 = 0$, so $\vec{v} = (-1 \quad 1)^T$. If $\lambda = 5$ it becomes $-v_1 + 4v_2 = 0$, so that $\vec{v} = (4 \quad 1)^T$. A general solution is therefore

$$\vec{x} = c_1\begin{pmatrix} -1 \\ 1 \end{pmatrix}e^{-5t} + c_2\begin{pmatrix} 4 \\ 1 \end{pmatrix}e^{5t}.$$

   The initial conditions give $-c_1 + 4c_2 = 7$ and $c_1 + c_2 = 3$, so $c_1 = 1$ and $c_2 = 2$. The solution is therefore

$$\vec{x} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}e^{-5t} + \begin{pmatrix} 8 \\ 2 \end{pmatrix}e^{5t}.$$

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3. What is an eigenvalue problem? What condition do we impose to find the eigenvalues of a matrix? (4 points)

Solution: An eigenvalue problem is the problem of finding a vector $\vec{v}$ and scalar λ such that, for a given matrix $\mathbf{A}$, $\mathbf{A}\vec{v} = \lambda\vec{v}$ (or, equivalently, $(\mathbf{A} - \lambda\mathbf{I})\vec{v} = \vec{0}$). To find the eigenvalues λ we require that $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$. ∎

a. How does the condition that you indicated in (3) follow from your statement of what an eigenvalue problem is? (4 points)

Solution: Because the eigenvalue problem $\mathbf{A}\vec{v} = \lambda\vec{v}$ can be rewritten as $\mathbf{A}\vec{v} = \lambda\mathbf{I}\vec{v}$, or $\mathbf{A}\vec{v} - \lambda\mathbf{I}\vec{v} = \vec{0}$, we can write it in the second of the forms given above: $(\mathbf{A} - \lambda\mathbf{I})\vec{v} = \vec{0}$. The condition is then just requiring that the determinant of the coefficient matrix of $\vec{v}$ be zero. ∎

b. Why do we impose the condition that you indicated in (3) (that is, what does imposing the condition guarantee us)? (4 points)

Solution: The requirement that the determinant be zero guarantees that the coefficient matrix $\mathbf{A} - \lambda\mathbf{I}$ is singular, which will result in the matrix equation $(\mathbf{A} - \lambda\mathbf{I})\vec{v} = \vec{0}$ having (an infinite number of) nontrivial solutions for $\vec{v}$. ∎

4. Consider the system $\vec{x}\,' = \begin{pmatrix} 2 & -5 \\ 4 & 6 \end{pmatrix} \vec{x}$. The eigenvalues and eigenvectors of the matrix $\begin{pmatrix} 2 & -5 \\ 4 & 6 \end{pmatrix}$ are $\lambda = 4 \pm 4i$ and $\vec{v} = \begin{pmatrix} -1 \pm 2i \\ 2 \end{pmatrix}$. Find the general (real-valued) solution to this problem. (12 points)

Solution: To write a real-valued general solution we need two real-valued linearly-independent solutions to the problem, which we can get by pulling apart the real and imaginary parts of a complex-valued solution. Using the plus in the eigenvalues and eigenvectors provided, a complex-valued solution is

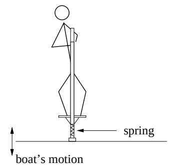
$$\vec{x} = \begin{pmatrix} -1 + 2i \\ 2 \end{pmatrix} e^{(4+4i)t} = e^{4t} \begin{pmatrix} -1 + 2i \\ 2 \end{pmatrix} (\cos(4t) + i \sin(4t))$$
$$= e^{4t} \left[\begin{pmatrix} -\cos(4t) - 2\sin(4t) \\ 2\cos(4t) \end{pmatrix} + i \begin{pmatrix} 2\cos(4t) - \sin(4t) \\ 2\sin(4t) \end{pmatrix} \right].$$

Thus a real-valued solution is

$$\vec{x}_g = c_1 \begin{pmatrix} -\cos(4t) - 2\sin(4t) \\ 2\cos(4t) \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} 2\cos(4t) - \sin(4t) \\ 2\sin(4t) \end{pmatrix} e^{4t}.$$

∎

5. A clown stands on a pogo stick (note: a pogo stick is essentially a spring; see figure to the right) on the deck of a boat. The up-and-down motion of the boat is periodic with a frequency ω. The clown's vertical displacement is then governed by the initial value problem



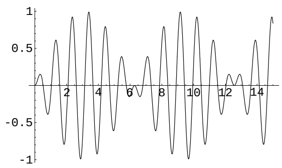
$$y'' = -cy' - 20y + 17\sin(\omega t), \quad y(0) = 0, \quad y'(0) = 0.$$

a. Is it possible that this equation will allow the the clown's displacement to exhibit beats? Explain any conditions that would have to be true for this to happen. (4 points)

Solution: This is possible if $c = 0$ (there is no damping) and $\omega \approx \omega_0$, where ω_0 is the natural frequency at which the pogo stick tends to oscillate (which is $\omega_0 = \sqrt{20}$). ∎

b. Suppose, regardless of what you said in (5a), that the clown's motion exhibits beats. Sketch a graph illustrating this. Explain in a couple of sentences what it says about the clown's displacement (that is, what can you say about how the displacement changes with time? Is the displacement large or small? Growing of decaying?, etc.). (6 points)

Solution: A solution that exhibits beats would appear thus:



This says that the clown is bouncing up and down with a fairly high frequency, and with a varying amplitude. Initially the height of the bounces is small, but it then increases until s/he is bouncing up and down with quite a large amplitude. It then decreases again, and so on. At some point we anticipate that the clown will be quite sick. ∎

c. Now suppose that $c = 4$ and that the boat's motion continues with $\omega = 4$ for a long time. Solve the initial value problem (insofar as you need to) and find an expression describing the long-term behavior of the clown's displacement. (12 points)

Solution: If $c = 4$, we're solving the problem $y'' + 4y' + 20y = 17\sin(4t)$. To determine the long-term behavior, we need to know y_p, because the complementary homogeneous solution y_c will decay on account of the non-zero damping ($c = 4$). Guess $y_p = A\cos(4t) + B\sin(4t)$. Plugging into the equation, we get

$$\begin{aligned} -16A + 16B + 20A &= 4A + 16B = 0 \\ -16B - 16A + 20B &= -16A + 4B = 17 \end{aligned}$$

(from the coefficients of $\cos(4t)$ and $\sin(4t)$, respectively). Thus, multiplying the first by four and adding, $68B = 17$, so $B = 1/4$. Then $A = -1$. Thus in the long-term the clown's displacement is given by

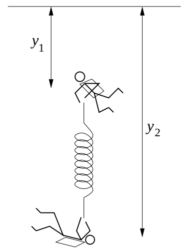
$$y_p = -\cos(4t) + \frac{1}{4}\sin(4t).$$

∎

6. Write down a differential equation for which you could use Variation of Parameters but *could not* use the Method of Undetermined Coefficients. Explain what characteristics of your equation make it a correct answer to this question. Be sure to include in your explanation what you use Variation of Parameters to find. *Note that you do not need to solve the problem that you write down.* (6 points)

   Solution: The two most obvious candidates for such an equation are things like $y'' + 4y = \tan(x)$ and $\frac{1}{x^2}y'' + y = e^{-2x}$. Variation of Parameters (and the Method of Undetermined Coefficients) works only for linear equations, which these are. The Method of Undetermined Coefficients (MUC) further requires that the forcing function $f(x)$ ($= \tan(x)$ or e^{-2x} in the above) be "nice"—that is, have a finite number of linearly independent derivatives. $f(x) = \tan(x)$ does not have this property, so MUC cannot be used for the first equation. MUC also relies on the differential equation we're solving having constant-coefficients, which isn't the case in the second equation. Therefore it could not be used in the second case even though the forcing $f(x) = e^{-2x}$ is very "nice." Finally, both Variation of Parameters and MUC determine a particular solution y_p for us. ■

7. Two skydivers, after leaping one-after-the-other from a plane, are playing with a long spring (figure to the right). The spring's equilibrium length is L. A system modeling this is



$$m_1\, y_1'' = m_1 g - c y_1' + k(y_2 - y_1 - L)$$
$$m_2\, y_2'' = m_2 g - c y_2' - k(y_2 - y_1 - L).$$

   a. Explain what each of the terms in the system represent, and therefore why it is a good model for this situation. (6 points)

   Solution: Going through the terms in the equations in order, $m_{1,2}y_{1,2}''$ are the inertial terms (mass times acceleration) which show up in Newton's law. Thus these are the one side of the equation $F = ma$, and the remaining terms are the sum of the forces acting on either skydiver. $m_{1,2}g$ is the force of gravity on either skydiver, and $-cy_{1,2}'$ is the force of air resistance. Air resistance acts in the opposite direction to motion, resulting in the negative sign. $k(y_2 - y_1 - L)$ is the spring force, which is proportional to the stretch on the spring, $y_2 - y_1 - L$. The spring pulls skydiver 1 down (in the positive direction) and skydiver 2 up (in the negative direction), accounting for the difference in signs given in the equation. Thus the system is a nice force equals mass times acceleration model for the two skydivers. ■

   b. Rewrite this as a *first-order* system of differential equations. (6 points)

   Solution: Let $x_1 = y_1$, $x_2 = y_1'$, $x_3 = y_2$, and $x_4 = y_2'$. Then

$$\begin{aligned}
x_1' &= x_2 \\
x_2' &= -kx_1/m_1 - cx_2/m_1 + kx_3/m_1 + g - kL/m_1 \\
x_3' &= x_4 \\
x_4' &= kx_1/m_2 - cx_3/m_2 - kx_3/m_2 + g + kL/m_2
\end{aligned}$$

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   c. Rewrite your system as a matrix equation. (4 points)

   Solution:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -k/m_1 & -c/m_1 & k/m_1 & 0 \\ 0 & 0 & 0 & 1 \\ k/m_2 & 0 & -k/m_2 & -c/m_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \begin{pmatrix} 0 \\ g - kL/m_1 \\ 0 \\ g + kL/m_2 \end{pmatrix}$$

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