LAB 1: CANCER, SERIES, AND ODE SOLUTIONS, PART B

1. Matlab

MATLAB commands we use in this lab are `ode45` and an add-on function `seriesapprox_w19`, as well as `plot` to plot solutions and `save` to save variables for future use. These are (re)introduced below. The lab assignment follows this section.

1.1. **ode45.** Finds a numerical approximation to a differential equation.

```
>> [tsol,ysol] = ode45( func_handle, [tmin,tmax], initial_v );
```

1.2. **plot.** Plot one vector against another.

```
>> plot( t_values, y_values );
```

1.3. **axis.** Set the x and y domains of a plot:

```
>> axis( [xmin xmax ymin ymax] );
```

1.4. **hold.** Plot later plots on the same axes (`hold on`), or replace the current plot (`hold off`):

```
>> hold on;
>> hold off;
```

1.5. **title, xlabel, ylabel.** Set the title, x- and y-axis labels for an existing plot:

```
>> title( 'title text' );
>> xlabel( 'x-axis label text' );
>> ylabel( 'y-axis label text' );
```

1.6. **legend.** Set the legend for the plot:

```
>> legend( 'label 1', 'label 2', 'label 3'...);
```

2. Background

In this lab, we are studying the Gompertz equation, a first-order ordinary differential equation which models the growth of cancerous tumors,

$$\frac{dy}{dt} = ry \ln(K/y). \tag{1}$$

The constants r and K in this equation are positive, and we consider $r = 0.1$ and $K = 10$. The function $y(t)$ gives the volume of the tumor at time t. The initial condition, $y(0) = y_0$, must be positive (that is, greater than zero), and we will in general take $y(0) = 1$.

You will complete a lab report as described in section 6 with your partner. This is due at the beginning of the next lab period of the following Lab.

3. Exercise 1

In your written homework you found the exact solution to the Gompertz equation. Check that you and your partner agree on the solution. If you haven't already, also find the exact solution to the $n = 1$ (linear) approximation to the differential equation.

Take $r = 0.1$, $K = 10$, and $y(0) = 1$. Plot the exact solution to the Gompertz equation, (1), with the exact solution to the linear approximation and a numerical solution (generated with `ode45`) of the cubic ($n = 3$) approximation (why do we omit the $n = 2$ approximation?).

How do the different solutions differ? How and where are they similar? Compare with your work in Part A, Exercise 3; does your work here give you confidence in the accuracy of the approximations to solutions of a differential equation that are generated by `ode45`?

4. Exercise 2

For all of the preceding work we have taken $y(0) = 1$. Would you expect the approximations to the Gompertz equation by expanding it around $y = K$ to be good approximations when $y = 1$? (Your work from Part A, Exercise 3 may shed some light on this.)

Let's consider some values close to the expansion point. Find the exact solution to the Gompertz equation and the linear approximation when $y(0) = 8$ (you may take $r = 0.1$ and $K = 10$ still). Find numerical approximations using `ode45` to the approximations with $n = 2$ and $n = 3$. Plot all of these solutions together. To the solutions to the approximate equations look similar to the exact solution? Do the different approximations behave as you expect?

Can you find an exact solution for the equation obtained by the $n = 2$ truncation of the Taylor series? This is a *logistic equation*, which we consider in [BB, §2.5]. It's a good challenge practice problem to find the exact solution!

5. Exercise 3

Finally, recall in the prelab we looked at the expansion of the differential equation around the point $y = 1$. Check with your partner that you agree on the form of the expansion in this case. Then use `ode45` to generate (approximate) solutions to the Gompertz equation and the $n = 1$, $n = 2$, and $n = 3$ Taylor approximations to the equation (you may need to pick carefully the time interval on which you are generating the solution) and plot them. How good are the approximations? What happens to the agreement between the solutions to the approximate equations and the solution to the Gompertz equation as time goes on? Be sure you can explain why this makes sense given what your work in Exercise 2 suggests about how well the solutions to the approximate equations give insight on the behavior of the original.

6. Lab Report

Imagine that you are a biomedical engineering consultant, and that a medical researcher has contacted you to give her insight on the development of cancer tumors. She is, in particular, interested in knowing what the model predicts for the behavior of the tumor in certain treatment regimes, and when different approximations to the model may be appropriate (and when they may be less so). Your writeup will be collaboratively produced by you and your partner, and both of you will submit the writeup paper. Note that you will need to include figures from the work that you did in the course of Parts A and B of the lab to produce a good writeup.

The questions that the research has posed are the following:

- If a treatment reduces the rate of tumor growth, will that have a significant impact on the long-term outcome of the cancer?
- What is the predicted long-term behavior of the tumor, and would this be altered if the initial tumor size was changed, e.g., by a surgical intervention that removed most of the tumor?
- Is a simplified form of the Gompertz model adequate to predict the behavior of the tumor, and are there circumstances in which the simplification would be significantly better or worse?
- If the model is simplified by assuming a small tumor size, what can (and can not) be determined from the resulting simplified model?

Your lab report should have the following format.

I. **Introduction:** Summarize the purpose and contents of your report in 3–6 complete sentences. You should include the Gompertz equation and the other equations you consider, with a note on how you analyze them, but otherwise keep the technical notation to a minimum.

II. **Body:** In the body of the report, you want to answer the questions posed by the researcher, by describing how the different parts of your analysis allow you to do this. In particular, all of the following should appear as you frame your answers.

   a. You should explain how your work in math 216 allows you to analyze the Gompertz equation. You will want to note what the equilibrium solutions are for the Gompertz equation, and what your analysis says about how the different parameters in the equation (r and K) affect the solution.

   b. You should explain how Taylor expansions are used to simplify the Gompertz equation, and when the simplifications are likely to be good approximations to the original model.

   c. And you should explain what the approximation obtained in Exercise 3 tells about solutions to the Gompertz equation, and the circumstances under which it might be useful.

III. **Conclusion:** Provide a short, several paragraph, summary of your results that ties together the work you have described in the body.

References

[BB] Brannan, James R, and William E Boyce. Differential Equations: an Introduction to Modern Methods And Applications. Third edition. Hoboken, NJ: Wiley, 2015.