LAB 1: CANCER, SERIES, AND ODE SOLUTIONS

1. MODEL AND OBJECTIVES

1.1. **Model.** In this lab, we will study the Gompertz equation, a first-order ordinary differential equation which models the growth of cancerous tumors:

$$(1) \qquad \frac{dy}{dt} = r\, y\, \ln(K/y).$$

The constants r and K in this equation are positive. The function $y(t)$ models the cell population of the tumor at time t.

1.2. **Objectives.** In this lab our goals are to see some connections with the material we've seen in calculus, understand how these may allow us to explore the behavior of solutions to nonlinear differential equations. In particular, we want to:

- learn how we can approximate a nonlinear ordinary differential equation (ODE) with a simpler (usually linear) ODE by using a Taylor polynomial (the truncation of a Taylor series) to approximate the nonlinear terms,[1] and

  > If you don't remember Taylor series, don't worry! We (re)introduce them to you in this prelab.

- build a sense of how solutions to linear equations behave, and how this appears in nonlinear equations.

In addition, we will see in passing how Taylor series and Taylor polynomials can be used to approximate the solutions to an ODE.

2. PRE-LAB

2.1. **Taylor series.** In our calculus classes, we learned how to construct Taylor series: that is, for a function $f(x)$ which has derivatives of all orders at a point x_0, we found a series of the form

$$(2) \qquad a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots = \sum_{n=0}^{\infty} a_n(x - x_0)^n = f(x).$$

The series on the left-hand side we call the Taylor expansion of $f(x)$ near x_0. Note that (2) is an equality, which means that $f(x)$ and the series are the *same function* in some neighborhood of x_0.

> "Neighborhood" means an interval around x_0 on which the series converges.

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[1]This is an introduction to a fundamental idea we will be seeing in class throughout the rest of the semester: that we can gain a qualitative understanding of nonlinear equations locally, by linearization.

Example 1*: The function* $\sin x$ *is equal to the following infinite series:*

$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1},$$

and the equality holds for all real numbers x*. What is* x_0 *in this case? What are the coefficients* a_n*? In particular, what is* a_2*?*
The expansion is in terms of $(x-0)^k$*, so* $x_0 = 0$*. The coefficients* a_n *are zero for all even* n *(and* $n = 0$*), and are* $\pm 1/n!$ *for odd coefficients. The first,* a_1*, is positive, and subsequent terms alternate sign.*

We can see how the series generates the function as we add terms to the sum; this is shown in the figure, below.
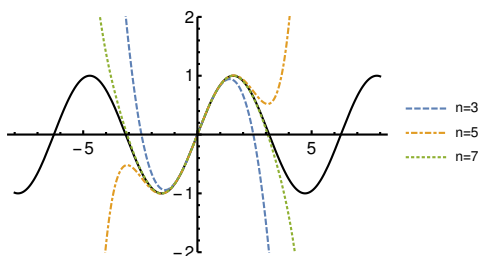
We see that the Taylor polynomials obtained by truncating the series to different values of n resemble the sine function better and better; if we continued to an infinite number of terms, we would have exact equality.

A consequence of the equality (2) that will be significant in this lab is that all derivatives of $f(x)$ and of the series are also equal as functions—in particular, they must be equal at the point $x = x_0$. This fact can be used to calculate the coefficients a_n, as we show in the following example.

Example 2*: How are the values of* a_n *related to* f *and its derivatives?*
Note that if we plug $x = x_0$ *into both sides of (2) we are left with* $f(x_0) = a_0$ *(this tells us the value of* a_0*!). Taking the derivative of each side of (2), we have*

$$f'(x) = 0 + a_1 + 2a_2(x - x_0) + 3a_3(x - x_0)^2 + \cdots$$

$$= \sum_{n=1}^{\infty} n\, a_n\, (x - x_0)^{n-1},$$

so that $f'(x_0) = a_1$*, that is,* $a_1 = f'(x_0)$*. Continuing with the second derivative, we have*

$$f''(x) = 2a_2 + 6a_3(x - x_0) + \cdots = \sum_{n=2}^{\infty} n(n-1)\, a_n\, (x - x_0)^{n-2},$$

and $f''(x_0) = 2a_2$*—so* $a_2 = \frac{1}{2} f''(x_0)$*. Another derivative gives us* $a_3 = \frac{1}{6} f'''(x_0) = \frac{1}{3!} f'''(x_0)$*. What happens as we continue taking derivatives?*

Each derivative pulls down another factor from the exponent, so that after k derivatives we have $a_k = \frac{1}{k!} f^{(k)}(x_0)$.

Thus, Example 2 tells us the coefficients of a Taylor series of a known function f, and thus that the series (2) is

$$(3) \qquad f(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x_0)(x - x_0)^n.$$

In practice, of course, it's difficult to work with a series which has an infinite number of terms, so we will often truncate the series to some number of terms and use that as an approximation to the function we want. In particular, as we will see repeatedly in this course, *linear approximations* often allow us to work with problems that would otherwise be intractable.

Exercise 1: The Taylor series for $\ln(y)$ about $y = 1$ is

$$(4) \qquad \ln(y) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(y - 1)^n$$

for $y - 1 \in (-1, 1]$ (that is, $y \in (0, 2]$). What polynomials do we get if we truncate this series at $n = 1$? $n = 2$? $n = 0$ (hint: the $n = 0$th approximation *is* defined!)? Compare the value of each of these with that of $\ln(y)$ at $y = 1.1$ and $y = 1.75$. Note how the error differs at the different y values.

We call the approximations we found in Exercise 1 *Taylor polynomials*, for obvious reasons. When we truncate the Taylor series to a polynomial of degree n, we say the approximation is "of order n," or "nth order."

2.2. **Approximating nonlinear differential equations.** At various times, we use Taylor polynomials to find approximations to nonlinear differential equations. This is a powerful tool, because linear equations are (in general) easy (or easier) to solve, and the insight we gain from the linear problem is usually applicable to the nonlinear system when the approximation is a reasonable one.

In particular, we will look for an approximation to our differential equations near *critical points*. We explore this idea in Exercises 2–4:

> We discuss critical points in [BB, §§1.2,2.5]. They are constant solutions, which we also call equilibrium solutions.

Exercise 2: Find the critical points of the Gompertz equation (1), by noting that at a critical point $y' = 0$ and solving. *(Is $y = 0$ a critical point? Does it solve the algebraic equation you get?)*

Once we have critical points, we will look for an approximation to the original equation by expanding nonlinearities using their Taylor series about the critical point.

Exercise 3: Find the first four terms of the Taylor series for $\ln(\frac{K}{y})$ about $y = K$ by using the formula we found in Example 2, (3).

Now we have done the hard part: we have a way to approximate the log term as a polynomial! Notice that there's one more piece that we need to resolve: we have now $y' = r\,y \sum_{n=1}^{\infty} a_n (y - K)^n$, where the a_n are determined by your work in Exercise 3, but note that to make the entire right-hand side into a Taylor series we need to rewrite $y = K + (y - k)$. This will ensure that all terms in our expression that involve y will be in terms of $y - K$, which is important for the approximation to be useful.

> **Exercise 4:** Plug in your expression for the expansion of $\ln(\frac{K}{y})$ into the equation $y' = r\,(K + (y - K))\ln(\frac{K}{y})$. Find approximations to the equation of order $n = 0$, $n = 1$, $n = 2$, and $n = 3$ (that is, find equations that truncate the resulting expression at the constant, linear, quadratic, or cubit terms in the expansion).

In general, we will use this technique to obtain a *linear equation*, which we can then solve. The solution that we obtain to this simpler equation will be a good approximation to the solution to the original equation near the expansion point x_0 (here, $y_0 = K$), as suggested by our work in Exercise 1.

Finally, note where you would expect an approximation obtained by the truncation that you did in Exercise 4 to be valid. We omitted higher-order terms in $(y - K)^n$, so we want those terms to be small for the approximation to be valid: that is, we want y to be close to K. If y is not close to K, we would expect the approximation not to be a good one.

Do we have to expand near a critical point? No! However, critical points are usually where interesting behavior of the system is captured. We can do the same analysis near other points, however, and explore this briefly in the following exercise.

> **Exercise 5:** By writing $y = 1 + (y - 1)$ and expanding $\ln(y)$ about $y = 1$ (using the Taylor series from Exercise 1, equation (4)), find order $n = 0$, $n = 1$, and $n = 2$ approximations to the Gompertz equation at $y = 1$.

2.3. **Series solutions to differential equations.** Another way we can use Taylor series is to look for a series that is the solution to a differential equation. For example, if we have $y' = c\,y$, with $y(0) = 1$, then we are saying there is a function $y(t)$ that satisfies this equation and initial condition. If this function has a convergent Taylor series $y(t) = \sum_{n=0}^{\infty} a_n t^n$ in a neighborhood of the initial condition, we should be able to follow the steps used in Example 2 to find the a_n and thus find the solution as a Taylor series. This is largely an exercise in bookkeeping, but it can sometimes provide insight on the behavior of the solutions to a differential equation.

<div align="center">REFERENCES</div>

[BB] Brannan, James R, and William E Boyce. Differential Equations: an Introduction to Modern Methods And Applications. Third edition. Hoboken, NJ: Wiley, 2015.