LAB 2: FIRST ORDER SYSTEMS AND THE VAN DER POL OSCILLATOR

1. Model and Objectives

1.1. **Model.** A van der Pol oscillator is a model of an active RLC circuit with a nonlinear resistor that dissipates energy when the amplitude of the current is high, and pumps energy into the system whenever the amplitude of the current is too low. The behavior of the resistor is described by a nonlinear function $f(x)$, where x is dimensionless current. As a function of time, the current $x(t)$ satisfies the second order differential equation

$$x'' + \mu \frac{df}{dx} x' + x = 0,$$

If you haven't worked with a second-order differential equation like this before, or haven't explored systems of differential equations, that's ok! We'll develop what we need here.

where μ is a positive constant. In this lab, we will take $f(x) = \frac{1}{3}x^3 - x$ so that the resulting equation is

$$x'' + \mu(x^2 - 1)x' + x = 0. \tag{1}$$

Note that the effect of the nonlinear resistor is modeled by the x' term.[1] For the duration of the lab, when we refer to "the" van der Pol equation, we will mean equation (1).

1.2. **Objectives.** In this lab our goals are to explore some of the fundamental links between higher-order differential equations and systems, and to see how linear and nonlinear systems are related and different. In particular, we want to

- See how to convert a second order ODE into a system of first order ODE's.
- See how we can linearize a nonlinear first order system to get a (solvable!) linear system, as we did in Lab 1.
- Compare local and global behavior of a system and see what the linearization of the system tells us about the behavior of the nonlinear system, and what it is unable to determine.

[1] We can see this from the equation by rewriting it as $x'' = \mu(1-x^2)x' - x$. The resistance (friction) term is the x' term, and if $|x| < 1$ the term is positive—adding energy to the system—and if $|x| > 1$ it is negative—reducing energy. A normal resistor would give a term like kx', with $k < 0$, and would always reduce the energy in the system.

2. Pre-Lab

2.1. **First Order Systems and Second Order Equations.** We can convert a given a linear second order equation

See [BB, §3.2 (p. 140)] to review the relationship between second order equations and first order systems! We cover all of this in more detail in chapter 4.

$$x'' = p(t)x' + q(t)x + g(t) \tag{2}$$

into a first order system by defining a new function $y(t) = x'(t)$. Then $y'(t) = x''(t)$, so y must satisfy the first order equation $y' = p(t)y + q(t)x + g(t)$. Therefore, if x is any solution to the second order equation, the pair of functions x and y must satisfy the first order system

$$\begin{aligned} x' &= y \\ y' &= p(t)y + q(t)x + g(t) \end{aligned} \tag{3}$$

Conversely, if x and y satisfy this system of equations, then x is a solution to the second order equation (2).

We will often want to write the system (3) in terms of a vector, $\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$. Note that this just takes the dependent variables in our system, x and y, and uses them as the components of the two element vector $\mathbf{x}$. If we similarly define $\mathbf{b}(t) = \begin{pmatrix} 0 \\ g(t) \end{pmatrix}$ and the matrix $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ q(t) & p(t) \end{pmatrix}$, we may express system (3) in the form $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{b}$.

Exercise 1: In a RLC circuit with constant resistance R, inductance L, and capacitance C, we denote the total charge on the capacitor at time t by $q(t)$. If there is no driving voltage applied to the circuit, the charge satisfies the linear second order equation

$$Lq'' + Rq' + \frac{1}{C}q = 0.$$

For a picture of an RLC circuit and derivation of this model see p. 213 of [BB].

(Note that this is of the same form as the van der Pol equation, (1), when $f'(x) = 1$, $R = \mu L$ and $C = 1/L$.) The current in the circuit, $u(t)$, is the derivative of $q(t)$ (that is, $u(t) = q'(t)$). Write down a first order system for q and u, in matrix form, which is equivalent to the second order equation for the RLC circuit.

We can likewise convert nonlinear second order equations, such as the van der Pol equation (1), into first order systems, but cannot write the resulting system in the form $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{b}$.

Exercise 2: Write the van der Pol oscillator (1) as a first order system in x and $y = x'$ (that is, a system like (3)). Explain why we cannot write it in the form $\mathbf{x}' = \mathbf{A}\mathbf{x}$ with $\mathbf{A}$ being a matrix with entries that do not depend on x or y.

2.2. **Linearization.** Even though nonlinear homogeneous systems cannot be written in the form $\mathbf{x}' = \mathbf{A}\mathbf{x}$, still we will be able to use linear equations to predict the behavior of solutions near the critical points of a nonlinear system. Critical points of an autonomous system

$$\begin{aligned} x' &= f(x, y) \\ y' &= g(x, y) \end{aligned}$$

are the pairs (x, y) such that both $f(x, y) = 0$ and $g(x, y) = 0$—so that they are the equilibrium solutions to the system (see [BB, §3.6 (p. 191)]).

Exercise 3: The motion of a pendulum (without damping) can be described by the nonlinear second order equation $\theta'' + \omega^2 \sin\theta = 0$, where θ is the angle from vertical and ω is a positive constant. Convert this equation to a first order system and find the critical points. Let $x = \theta$ and $y = \theta'$ in your system.

In Lab 1, one of the techniques we used was to approximate solutions and differential equations by replacing a nonpolynomial function $f(x)$ with truncations of its Taylor series centered around some point x_0. The first order truncation gives the best linear approximation to $f(x)$ at x_0.

This "linear approximation" to $f(x)$ is, in fact, the tangent line to $f(x)$ through $(x_0, f(x_0))$. Try checking this for yourself by calculating the tangent line explicitly. Remember that the equation of a line is $y = mx + b$, and that m and b can be determined uniquely by the slope of the line and a single point on the line.

To study the behavior of solutions in a neighborhood of a critical point of a nonlinear system, we replacing the nonlinear functions in the system with their linear approximations around the critical points. This is called the "linearization" of the system at that critical point.[2] Linearization gives the mathematical description of the behavior seen near critical points. We will learn about this topic in more depth in [BB, §3.6] and [BB, §7.2].

Example 1*: Linearize the nonlinear pendulum of Exercise 3 at the critical point* $(0, 0)$.

The only nonlinear function in the system is $\sin x$, which we will replace with the first order truncation of its Taylor series around $x = 0$ (because that's the critical point). Recall that the Taylor series of $\sin x$, expanded around $x = 0$, is

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

and the linear truncation is simply $\sin x \approx x$. Therefore, the linearization of your solution to Exercise 3 near $(0, 0)$ is

$$\begin{aligned} x' &= y \\ y' &= -\omega^2 x \end{aligned} \tag{4}$$

[2] In this lab, we see only examples where the nonlinear parts of the system are functions of x alone. For more general systems, we will have to learn how to construct Taylor series of functions of two variables—we will do this in [BB, §7.2].

Turning now to the van der Pol system, the only critical point is $(0, 0)$, and since the system consists only of polynomials, the linearization at $(0, 0)$ is obtained simply by dropping the nonlinear terms from the right hand side of the equations:

$$\begin{aligned} x' &= y \\ y' &= -x + \mu y. \end{aligned} \tag{5}$$

In matrix form we write this as $\mathbf{x}' = \left(\begin{smallmatrix} 0 & 1 \\ -1 & \mu \end{smallmatrix}\right) \mathbf{x}$. We will use both (5) and the original van der Pol equation (1) (and its formulation as a system, which you found in Exercise 2) in the rest of this lab.

References

[BB] Brannan, James R, and William E Boyce. Differential Equations: an Introduction to Modern Methods And Applications. Third edition. Hoboken, NJ: Wiley, 2015.