# LAB 3: LASERS, LINEAR SYSTEMS, AND HARMONIC OSCILLATORS, PART A



## 1. Introduction

Recall that before working each part of the lab you should read through it. Both of Parts A and B have a first section that describes the *MATLAB* commands that we will be using in the lab. Read through those quickly, so that you know what they are, and remember to refer back to this section as you work the lab for help with *MATLAB*.

Next, we give an overview of the model that we introduced in the Prelab, and introduce the question that you will be answering in your lab report. The remainder of each part of the lab are the exercises that constitute the work that you will need to complete the lab report. The actual lab report assignment is given at the end of the Part B document.

You will complete all of the work for this lab **in pairs**, with a partner. You will write your lab report together, and submit just one copy of that.

## 2. Matlab

*MATLAB* commands we use in this lab are `eig`, `ode45` and `plot`.

### 2.1. eig.

This *MATLAB* command calculates the eigenvectors and eigenvalues for a matrix:

```
>> [ evecs, evals ] = eig( mat )
```

The returned variables `evecs` and `evals` are both matrices. The columns of `evecs` are the eigenvectors of the matrix `mat`, and the diagonal entries of `evals` are the corresponding eigenvalues. The matrix $\begin{pmatrix} 1 & 5 \\ 2 & 4 \end{pmatrix}$ has eigenvalues $\lambda_{1,2} = -1, 6$, with eigenvectors $\mathbf{v}_1 = \begin{pmatrix} -5 & 2 \end{pmatrix}^T$ and $\mathbf{v}_2 = \begin{pmatrix} 1 & 1 \end{pmatrix}^T$, so that

```
>> [ evecs, evals ] = eig( [ 1 5; 2 4 ] )
```

returns `evecs = [-0.928 -0.707; 0.371 -0.707]` and `evals = [-1 0; 0 6]`. Note that *MATLAB* normalizes eigenvectors to have unit length; to get vectors with unit entries, divide by the smallest value, e.g.,

```
>>  evec1 = evecs(:,1)/evecs(2,1)
```

makes `evec1` be the first column of the matrix `evecs` (recall indexing is `(row, column)`), with each entry divided by the first entry in the second row. The result is `evec1 = [-2.5; 1]`! Finally, note that if you just type

```
>>  eig( [ 1 5; 2 4 ] )
```

the returned value is a column vector of just the eigenvalues; thus we'd get `ans = [-1; 6]`.

The notation $^T$ indicates the *transpose* of a matrix or vector, which just swaps the rows and columns of the matrix or vector. Thus the transpose of a row vector is a column vector! We use transposes here as a formatting trick to avoid wide line spacing around the vectors.

2.2. **format.** Set the output format for *MATLAB* calculations. `format long` results in answers being shown to 15 digits after the decimal point; `format`, or `format short`, shows four digits.

2.3. **ode45.** Finds a numerical approximation to a differential equation or system of equations:

```
>> [tsol,xsol] = ode45(@(t,x) [rhs1;rhs2], [tmin,tmax], [x0;y0]);
```

For $\mathbf{x}' = \mathbf{A}\mathbf{x}$, if we've defined the matrix `amat` to be $\mathbf{A}$, we can use

```
>> [tsol,xsol] = ode45(@(t,x) amat*x, [tmin,tmax], [x0;y0]);
```

2.4. **plot.** Plot one vector against another; e.g., to plot component plots from output from `ode45`,

```
>> plot( tsol, xsol(:,1), '-k', tsol, xsol(:,2), '--k' );
```

## 3. Background

In this lab we are considering a model for a laser; with $N$ giving the *population inversion* (difference between the number of atoms in a higher energy state and in the base, ground-level state) and $P$ the intensity (scaled number of photons),

$$\begin{aligned} N' &= \gamma(A - N(1+P)) \\ P' &= P(N-1), \end{aligned} \tag{1}$$

where $\gamma$ and $A$ are positive constants.

We derived this system in the Pre-Lab; we describe the governing equations from each step of the derivation below. The atoms in the laser can be in one of three energy states, $E_1$, $E_2$ and $E_3$. In the absence of an energy source or lasing photons, the number of atoms in each energy state are given, with $\mathbf{n} = \begin{pmatrix} n_1 & n_2 & n_3 \end{pmatrix}^T$, by

$$\mathbf{n}' = \mathbf{G}\mathbf{n}, \tag{2}$$

where

$$\mathbf{G} = \begin{pmatrix} 0 & \gamma_{21} & \gamma_{31} \\ 0 & -\gamma_{21} & \gamma_{32} \\ 0 & 0 & -\gamma_{32} - \gamma_{31} \end{pmatrix}.$$

For ruby lasers, which we consider here, $\gamma_{21}^{-1} = 3$ ms and $\gamma_{32}^{-1} = \gamma_{31}^{-1} = 0.1$ $\mu$s.

To create the laser, we add an energy pump, which stimulates atoms to move between the ground and third energy states, so that

$$\mathbf{n}' = (\mathbf{W} + \mathbf{G})\mathbf{n} \tag{3}$$

where $\mathbf{W} = \begin{pmatrix} -W_p & 0 & W_p \\ 0 & 0 & 0 \\ W_p & 0 & -W_p \end{pmatrix}$.

Finally, we consider the number of photons in the system that can cause the release of additional photons through atoms changing from states $E_2$ to $E_1$; adding this effect and introducing the population inversion variable $n = n_2 - n_1$ leads to (1) after we rescale the variables.

## 4. Lab Report

In your lab report, you will write a paper that looks at the mathematics of the models developed in the Pre-Lab, and how they describe the functioning of the laser. In particular, you will examine

- What the models for the number of atoms in each energy state are, both without and with the energy pump, and how your eigenvalue analysis and solution plots illustrate the physical behavior of the system in either case.
- What additional effect the nonlinear system includes, what the equilibrium solutions of the system are, and what those suggest about the possible long-term behavior of the laser.
- How the stability of the different equilibrium solutions depends on the parameters in the problem, and what the linearization tells you about the stability and expected behavior or the nonlinear system.
- What the effect of a nonconstant parameter $A$ is on the laser's output intensity, how this is similar to the phenomenon of resonance, and how the characteristics of the output intensity in this case may or may not be desirable.

As you work through the lab, you will want to think about how the exercises you are doing provide insight on these aspects. The details of the report are described at the end of Part B of the lab.

## 5. Part A Exercises

Write out the matrices $\mathbf{G}$ and $\mathbf{W}$, using units of ms. Let $W_p = 100\ \text{ms}^{-1}$. Check that both you and your partner get the same thing. Also check that you got the same critical points in your Pre-Lab, Exercise 3.

**Exercise 1.** Use *MATLAB*'s `eig` command to find the eigenvalues and eigenvectors of the $3 \times 3$ matrix $\mathbf{G}$. Thinking about what the solution $\mathbf{n}$ will be to (2), what do the eigenvalues and eigenvectors tell you about the long-term behavior of the system? In particular, in what energy state do the atoms in the laser end up? Is this consistent with your expectations given what the system is modeling (recall from the Prelab that $\mathbf{G}$ captures the effect of spontaneous decay between energy states)? *(You may want to use `format long` when looking at the eigenvalues.)*

Next repeat your analysis for the materix $\mathbf{G} + \mathbf{W}$. How is result different?

**Exercise 2.** Numerically solve the system (2) using `ode45` and the initial condition $\mathbf{n}(0) = \begin{pmatrix} 1/3 & 1/3 & 1/3 \end{pmatrix}^T$. Use $0 \le t \le 10$. Plot your solutions for $n_1$, $n_2$, $n_3$, and the population inversion $n = n_2 - n_1$, against time. Compare your result with your prediction from exercise 1.

Then numerically solve the system (3) using `ode45` and the initial condition $\mathbf{n}(0) = \begin{pmatrix} 1/3 & 1/3 & 1/3 \end{pmatrix}^T$. Use $0 \le t \le 0.1$. Plot your solutions for $n_1$, $n_2$,

$n_3$, and the population inversion $n = n_2 - n_1$. Compare your result with your prediction from exercise 1, and the result for the system with only **G**, (2).

Is it reasonable to assume that $n_3(t) = 0$ for all $t$?

**Exercise 3.** Finally, numerically solve (using `ode45`) the system (1) with $A = 0.5$ and $\gamma = 0.01$, using the initial conditions $(N(0), P(0)) = (-0.01, 0.01)$. Based on your work in exercise 2, explain why the initial condition for $N$ might be reasonable. How is the behavior of $N$ here different from what you saw for $n$ in exercise 2? What effects are included in (1) that are not included in your work in exercises 1 and 2?

## References

[BB] Brannan, James R, and William E Boyce. *Differential Equations: an Introduction to Modern Methods And Applications.* Third edition. Hoboken, NJ: Wiley, 2015.

[EG] Erneux, Thomas, and Pierre Glorieux. *Laser Dynamics*. Cambridge: Cambridge University Press, 2010.