

LAB 3: LASERS, LINEAR SYSTEMS, AND HARMONIC OSCILLATORS, PART B

(c)2019 UM Math Dept
licensed under a Creative Commons
By-NC-SA 4.0 International License.

1. MATLAB

MATLAB commands we use in this lab include the following.

1.1. **eig.** Finds the eigenvectors and eigenvalues for a matrix.

```
>> [ evecors, evalues ] = eig( matrix )
```

1.2. **format.** Set the output format for *MATLAB* calculations.

```
>> format long
```

1.3. **ode45.** Finds a numerical approximation to a differential equation or system of equations; for a system the first argument returns a vector.

```
>> [tsol,xsol] = ode45(@(t,x) [rhs1;rhs2], [tmin,tmax], [x0;y0]);
```

1.4. **plot.** Plot one vector against another; e.g., to plot component plots from output from *ode45*,

```
>> plot( tsol, xsol(:,1), '-k', tsol, xsol(:,2), '--k' );
```

and a trajectory in the phase plane is given by

```
>> plot( xsol(:,1), xsol(:,2), '-k' );
```

2. BACKGROUND

In this lab we are considering a model for a laser; with N giving the population inversion (difference between the number of atoms in a higher energy state and in the base, ground-level state) and P the intensity (scaled number of photons),

$$(1) \quad \begin{aligned} N' &= \gamma(A - N(1 + P)) \\ P' &= P(N - 1), \end{aligned}$$

where γ and A are positive constants.

These governing equations follow from a model for the number of atoms in the laser material (ruby) that are in each of three energy states,

$$(2) \quad \mathbf{n}' = \mathbf{G}\mathbf{n},$$

which is modified by the addition of an energy pump that stimulates atoms to move between states E_1 (the ground state) and E_3 (the highest energy state), to get

$$(3) \quad \mathbf{n}' = (\mathbf{W} + \mathbf{G})\mathbf{n}.$$

In these,

$$\mathbf{G} = \begin{pmatrix} 0 & \gamma_{21} & \gamma_{31} \\ 0 & -\gamma_{21} & \gamma_{32} \\ 0 & 0 & -\gamma_{32} - \gamma_{31} \end{pmatrix} \quad \text{and} \quad \mathbf{W} = \begin{pmatrix} -W_p & 0 & W_p \\ 0 & 0 & 0 \\ W_p & 0 & -W_p \end{pmatrix}.$$

We take $\gamma_{21}^{-1} = 3 \text{ ms}$, $\gamma_{32}^{-1} = \gamma_{31}^{-1} = 0.1 \text{ } \mu\text{s}$, and $W_p = 100 \text{ ms}^{-1}$.

Considering the number of photons in the system that can cause the release of additional photons and rescaling variables appropriate leads to (1). You found the critical points of (1) in the Pre-Lab. Linearizing around $(N, P) = (1, A - 1)$ gives the system

$$(4) \quad \begin{aligned} u' &= -\gamma(Au + v) \\ v' &= (A - 1)u. \end{aligned}$$

In this part of the lab we look at what this linearization (valid when N and P are near 1 and $A - 1$, respectively) tells us about the behavior of the nonlinear system, (1), and generalize (1) to allow for a non-constant A .

You and your partner are responsible for completing a lab report as described in section 6, due at the beginning of Workday 1 of the following Lab.

3. EXERCISE 1

For this and the following exercises, use $\gamma = 0.05$, and take $(N(0), P(0)) = (0.01, 0.01)$.

In the Pre-Lab and Part A, Exercise 3, and possibly in the written homework, you considered the critical points of (1) and their stability. To further investigate their stability, generate graphs of the behavior of N and P in system (1) with $A = 0, 0.5, 1, 1.5$ and 3 . Note how the result illustrates a transition in the stability of the different critical points, and how this is consistent with your work in the Pre-Lab and Part A.

When do you see (decaying) oscillatory solutions? What does this tell you about the eigenvalues of the linear system (4)?

4. EXERCISE 2

Review your work in the Pre-Lab and, possibly, written homework, where you found the equations for and, possibly, solutions $v(t)$ and P_{RO} . Verify that you have the same result as your partner. If you haven't already, find the eigenvalues of (4) (using `eig` if you like), or the second order equation for v you found in the Pre-Lab, Exercise 4. Verify that this is consistent with your observations in Exercise 1, above.

Recall that P_{RO} is the solution for $v(t)$ (the intensity) in the linearized laser system. To get a more concrete sense of how P_{RO} is a good approximation to the nonlinear solution P , consider $\gamma = 0.05$ and $A = 3$. Plot, in different graphs, the solution $P(t)$ from system (1) and $P_{RO}(t)$. Notice that if you consider only $t > t_1$, for some value t_1 , the graph of $P(t)$ looks like the graph of P_{RO} . Note when you expect this to be—that is, for what values of P should P_{RO} resemble

P ? Plot both P and P_{RO} for larger values of t (why do we pick larger values of t ?) to show that the graphs are similar. How do they differ?

5. EXERCISE 3

Now suppose that the parameter A is no longer constant, and is instead given by $A(t) = A_0 + 2a \cos(\omega t)$.¹ We expect $a \ll 1$, so that this results in a slight oscillatory behavior in A . We will consider different values of ω in the following. To start with, plot $A(t)$ with some $A_0 > 1$, say, $A_0 = 3$, some $0 < a < 1$, say, $a = 0.25$, and a value of $\omega = \sqrt{\gamma(A_0 - 1)}$. How would you expect this A to change the intensity that you obtain from the laser? Solve (1) with this $A(t)$ using `ode45` to determine what happens to the intensity, and thus how close your expectation is.

It turns out that the amplitude of the oscillations in the long-term response behaves very similarly to the response of a sinusoidally forced linear equation like those we study in [BB, §4.6]. There we see that a *gain rate* gives the amplitude of the response relative to that of the forcing. For this problem, there is a similar scaling factor for the amplitude of the response,

$$(5) \quad g(\omega) = \frac{2a\gamma(A_0 - 1)}{\sqrt{(\gamma(A_0 - 1) - \omega^2)^2 + A_0^2\gamma^2\omega^2}}.$$

Plot $g(\omega)$ as a function of ω for your A_0 and a . Where does it have a maximum? You could find this by differentiating $g(\omega)$ and setting it to zero; we save you the effort and suggest that the result is $\omega = \sqrt{\gamma(A_0 - 1)}$. We will call this value ν . Calculate this value and verify that it seems correct.

Next plot solutions to (1) with $A(t)$, found with `ode45`, with $\omega < \nu$, $\omega = \nu$, and $\omega > \nu$. How does the long-term behavior of the solution change? What is true of the case $\omega = \nu$?

In [BB, §4.6] we further investigate the behavior of forced systems to see how a gain function like (5) may be found, and what it tells us about the response to the forced system!

6. LAB REPORT

Your biomedical engineering consultant job was so successful that you have been hired by a company that is building lasers. Suppose that you have been asked to write a report to a scientifically minded prospective customer explaining the behavior of the lasers modeled in this lab. In your report you will want to address:

- What the models for the number of atoms in each energy state are, both without and with the energy pump, and how your eigenvalue analysis and solution plots illustrate the physical behavior of the system in either case.
- What additional effect the nonlinear system includes, what the equilibrium solutions of the system are, and what those suggest about the possible long-term behavior of the laser.

¹Why might we want this? An example would be a in laser surgery, in which a pulsed laser is used to cut tissue.

- How the stability of the different equilibrium solutions depends on the parameters in the problem, and what the linearization tells you about the stability and expected behavior of the nonlinear system.
- What the effect of a nonconstant parameter A is on the laser's output intensity, how this is similar to the phenomenon of resonance, and how the characteristics of the output intensity in this case may or may not be desirable.

Your lab report should have the following format:

- I. **Introduction:** Summarize the purpose and contents of your report in 3–6 complete sentences. You should include the systems (3) and (1), indicating what the functions in the systems are and how they are related, but otherwise keep the technical notation to a minimum.
- II. **Body:** In the body of the report, you should address the points noted above. You will want to include relevant equations, calculations, and graphs. You should also explain how your work in math 216 allows you to analyze this system, in particular, (1) how our eigenvalue solution methods explain the behavior of (3) and the behavior of (1) near the equilibrium solution, and (2) how the effect of non-constant A is related to our understanding of sinusoidally forced second-order differential equations.
- III. **Conclusion:** Provide a short, several paragraph, summary of your results that ties together the work you have described in the body.

REFERENCES

- [BB] Brannan, James R, and William E Boyce. *Differential Equations: an Introduction to Modern Methods And Applications*. Third edition. Hoboken, NJ: Wiley, 2015.
- [EG] Erneux, Thomas, and Pierre Glorieux. *Laser Dynamics*. Cambridge: Cambridge University Press, 2010.