LAB 3: LASERS, LINEAR SYSTEMS, AND HARMONIC OSCILLATORS

1. Objectives and Instructions

1.1. **Model.** In this lab we are going to study a system of differential equations which models the excitation of atoms in a lasing medium. Essentially, we will be learning (some of) the mathematics behind how lasers work. Atomic physics is, of course, outside of the scope of this course, but the modeling equations are ones we are able to analyze.

The system we will be analyzing is

$$\begin{aligned} N' &= \gamma(A - N(1+P)) \\ P' &= P(N-1) \end{aligned} \tag{1}$$

where P is the laser's intensity and N is the population inversion function for atoms in the laser[1], and γ and A are constants. This is a nonlinear system which we cannot solve analytically, but we will see that for large t, the intensity P is similar to the solution of a second order equation for a damped vibration, and we will explore resonance in this context.

1.2. **Objectives.** Our goals for this lab are to extend our use of matrix methods and to further investigate the behavior of solutions to linear and nonlinear equations. In particular, we will:

- see relationships and differences between first-order linear and nonlinear systems; and
- investigate damped harmonic oscillations and resonance.

Note that the first of these looks at *systems* of equations, while to look at the forcing that gives resonance, as we do in the second, we will look at an equivalent *single, higher-order* differential equation. We have already seen that a single higher-order differential equation can be written as a system to provide insight on the differential equation; in this lab we also make the transformation of a system into an equivalent second-order differential equation to provide insight on the system.

The mathematics of laser dynamics is a huge subject! We obviously won't be doing it all here; what we need is developed in the course of the pre-lab, and we'll leave out some of the trickier details. In [EG] there's a lot more detail than we present here.

2. Pre-Lab

In this Pre-Lab, we will see in a general sense where (1) comes from. The three key things we need to know are: 1. that the atoms in the lasing medium (ruby, for our work here) can be in one of three energy states; 2. that we can add energy to the laser, so that we move atoms to a higher energy state; and

[1] What "intensity" and "population inversion" mean is explained in §2.2.

3. that atoms tend to decay to lower energy states, and the energy released by the decay can be manifest as a released photon—and when the released photons can trigger the release of more photons, we get "lasing," which is the release of coherent, laser light.

2.1. **Energy levels.** In a ruby laser, each atom of the ruby is in one of three energy states, which we will refer to as E_1, E_2, and E_3. E_1 is the lowest energy level (the ground state), and E_3 the highest energy. Let $n_1(t)$, $n_2(t)$, and $n_3(t)$ be the number of atoms in each energy level at time t. Atoms in the E_2 and E_3 states will naturally decay to a lower level by releasing energy, so that without external stimulation all atoms will eventually fall to the ground state E_1. The laser adds an "energy pump" that moves atoms from state E_1 to state E_3 and increases the tendency of atoms in state E_3 to decay to E_1. The combination of the decay and energy pump means that n_1, n_2 and n_3 satisfy the system

The notation $\mathbf{v}^T$ indicates the *transpose* of the vector $\mathbf{v}$. This is obtained by flipping rows and columns: thus, the transpose of a row vector is a column vector: $\begin{pmatrix}1 & 2 & 3\end{pmatrix}^T = \begin{pmatrix}1\\2\\3\end{pmatrix}$.

$$
\begin{aligned}
n_1' &= -W_p n_1 + \gamma_{21} n_2 + (\gamma_{31} + W_p) n_3 \\
n_2' &= -\gamma_{21} n_2 + \gamma_{32} n_3 \\
n_3' &= W_p n_1 - (\gamma_{32} + \gamma_{31} + W_p) n_3,
\end{aligned}
\tag{2}
$$

or, in matrix form with $\mathbf{n} = \begin{pmatrix}n_1 & n_2 & n_3\end{pmatrix}^T$,

$$
\begin{aligned}
\mathbf{n}' = (\mathbf{G} + \mathbf{W})\mathbf{n} &= \left(\begin{pmatrix} 0 & \gamma_{21} & \gamma_{31} \\ 0 & -\gamma_{21} & \gamma_{32} \\ 0 & 0 & -\gamma_{32} - \gamma_{31} \end{pmatrix} + \begin{pmatrix} -W_p & 0 & W_p \\ 0 & 0 & 0 \\ W_p & 0 & -W_p \end{pmatrix} \right) \mathbf{n} \\
&= \begin{pmatrix} -W_p & \gamma_{21} & \gamma_{31} + W_p \\ 0 & -\gamma_{21} & \gamma_{32} \\ W_p & 0 & -\gamma_{32} - \gamma_{31} - W_p \end{pmatrix} \mathbf{n}.
\end{aligned}
\tag{3}
$$

Here the matrix $\mathbf{G}$ captures the effect of spontaneous decay between energy states: the γ_{ij} are the (constant, positive) rates at which atoms spontaneously decay from level i to level j. And the matrix $\mathbf{W}$ gives the effect of the energy pump.

Example 1*: Why are the entries in the second row and column of* $\mathbf{W}$ *all zero?*
The matrix $\mathbf{W}$ models the effect of the energy pump, which moves atoms from state E_1 to state E_3 and stimulates the release of energy from atoms in state E_3 so that the drop to state E_1. It doesn't have any effect on atoms in state E_2. The second column gives the effect of atoms in state n_2 on those in states in n_1 and n_3, and the second row the change in n_2 as a result of the energy pump. Both of these need to be zero for the pump to be behaving as advertised.

Exercise 1: Why does the first column of G contain only zeros? Why do some γ_{ij} have a minus sign and the others a plus sign?

In the case of ruby lasers, the spontaneous emission rates are $\gamma_{21} = 1/(3 \text{ ms})$ and $\gamma_{32} = \gamma_{31} = 1/(0.1\ \mu\text{s})$. Observe that $\gamma_{21} < \gamma_{32}$, and, in fact, γ_{32} and γ_{31} are not just larger than γ_{21}, but much much much larger! Physically, this means is that atoms in energy level E_3 decay almost immediately to E_2 or E_1, and relative to this, atoms in level E_2 decay slowly to E_1. This discrepancy in behavior between the energy levels is necessary for lasers to exist at all.

Note that the units of each γ_{ij} are 1/time! Why is this? Look at the units of n' and n in (3) to see. The conversion between the units of the constants γ_{ij} is: 1μs $= 10^{-3}$ ms, so $\gamma_{31}^{-1} = 10^{-4} ms$.

We are also implicitly assuming that the decay from E_3 to E_2 is through expulsion of energy like heat or a vibration instead of by releasing a photon.

2.2. **Stimulated emission and lasers.** Finally, we need to introduce the photons that make the laser. (The word LASER is an acronym for "Light Amplification by Stimulated Emission of Radiation.") When an atom changes from a state E_2 to E_1 it releases energy, which may take the form of a photon. The "Amplification" in the "LASER" begins when photons that are emitted by this state change circulate through the lasing medium (the ruby), interact with another atom of energy E_2 and stimulate it to change to state E_1 and emit another photon of the same frequency. This is called "Lasing." Note that lasing has the effect of increasing the number of photons in the system, and decreasing the number of atoms in energy level E_2.

To model this, we rewrite the system (3) to account for this decrease, and add an equation for the number of lasing photons in the system. Letting $p(t)$ be this number of photons, the equation for p turns out to be

$$p' = p(-\gamma_c + K(n_2 - n_1)),$$

where K is a positive constant called the gain rate, which represents the increase in lasing photons because of the photons' interaction with atoms in state E_2 to stimulate release of more photons, and γ_c is the rate at which lasing photons leave the system entirely (as laser light!). Note that this equation is in terms of $n_2 - n_1$. We will call this the "population inversion function" in a moment.

To rewrite (3), note that because the atoms at energy level E_3 decay much faster than those at E_2 we might get away with the assumption that $n_3(t) = 0$. If we take $n_3(t) = 0$ and rewrite (3) in terms of the "population inversion function" $n(t) = n_2(t) - n_1(t)$, we can rewrite the system as a single equation (you do this in Exercise 2).

Example 2: Let n_T be the total number of atoms in the laser, so that $n_T = n_1 + n_2 + n_3$. Derive expressions for n_1 and n_2 in terms of n and n_T, assuming that $n_3(t) = 0$.
Because $n_3 = 0$, we have $n_T = n_1 + n_2$. By definition, $n = n_2 - n_1$. If we add these two expressions, we get $2n_2 = n_T + n$, so that $n_2 = \frac{1}{2}(n_T + n)$. If we subtract them, we get $2n_1 = n_T - n$, so that $n_1 = \frac{1}{2}(n_T - n)$.

Exercise 2: Rewrite system (3) as a single equation in n, by assuming that $n_3 = 0$ and subtracting the remaining equations to get an equation for $n' = (n_2 - n_1)'$. (You will need the results derived in Example 2.)

Finally, the lasing photons reduce the number of atoms in state E_2, so we have to add a term to the equation that you found in exercise 2, to get the nonlinear system

$$n' = -(\frac{1}{2}W_p + \gamma_{21})n + (\frac{1}{2}W_p - \gamma_{21})n_T - 2Knp$$
$$p' = p(-\gamma_c + Kn) \tag{4}$$

If you're curious you can see how this rewriting works by taking $n(t) = aN(T)$, $p(t) = bP(T)$, and $t = cT$; plugging in and picking the right values for a, b and c will give (5).

It is convenient to rewrite (4) so that time is measured as a number of photon decay periods, and so that the population inversion function and number of lasing photons are measured as fractions of various equilibrium values. We omit the details of how that is done; the resulting, simplified, system is that which we introduced as our model at the beginning of the lab,

$$N' = \gamma(A - N(1 + P))$$
$$P' = P(N - 1). \tag{5}$$

We refer to the scaled variable N as the population inversion, and call P the intensity function. The constants γ and A are combinations of the other constants in the problem (it happens that $\gamma = \frac{\frac{1}{2}W_p+\gamma_{21}}{\gamma_c}$ and $A = \frac{(\frac{1}{2}W_p-\gamma_{21})Kn_T}{(\frac{1}{2}W_p+\gamma_{21})\gamma_c}$). We can think of A as a measure of how efficient the laser is (that is, how much it intensity increases as the population inversion increases) and γ as a measure of how lasing photons build up in the laser (that is, a ratio of photon creation to photon release as laser light). We will assume that these are positive in all lab exercises.

Exercise 3: Find the critical points of (5).

Exercise 4: The linearization of (5) at $(1, A - 1)$ is

$$u' = -\gamma(Au + v), \quad v' = (A - 1)u.$$

(a) Rewrite this linear system as a single second order linear equation for v.
(b) What is the value of P when $v = 0$? How are P and v related?
(c) The function $P_{RO}(t) = v(t) + A - 1$ is called the relaxation oscillation (RO) of the laser. How is it related to P? Use your solution to part (a) to write down a second order equation which has P_{RO} as its solution.

References

[BB] Brannan, James R, and William E Boyce. *Differential Equations: an Introduction to Modern Methods And Applications.* Third edition. Hoboken, NJ: Wiley, 2015.

[EG] Erneux, Thomas, and Pierre Glorieux. *Laser Dynamics.* Cambridge: Cambridge University Press, 2010.