

LAB 5: THE LORENZ SYSTEM AND WEATHER PATTERNS, PART A

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1. INTRODUCTION

Recall that before working each part of the lab you should read through it. Both of Parts A and B have a first section that describes the *MATLAB* commands that we will be using in the lab. Read through those quickly, so that you know what they are, and remember to refer back to this section as you work the lab for help with *MATLAB*.

Next, we give an overview of the model that we introduced in the Prelab, and introduce the questions that you will be answering in your lab report. The remainder of each part of the lab are the exercises that constitute the first part of the work that you will need to complete the lab report. The actual lab report assignment is given at the end of the Part B document.

You will complete all of the work for this lab **in pairs**, with a partner. You will write your lab report together, and submit the same report.

2. MATLAB

MATLAB commands we use in this lab include the following.

- 2.1. **disp**. Displays text to the command window. For example,

```
>> disp('This is text sent to the command window')
```
- 2.2. **ode45**. Finds a numerical approximation to a differential equation or system of equations:

```
>> [tsol,xsol] = ode45(f_handle, [tmin tmax], init_cond);
```
- 2.3. **plot**. Plot one vector against another; e.g.,

```
>> plot( tsol, xsol(:,1) );
```
- 2.4. **plot3**. Plot a three-dimensional figure; input are a vector of x -values, a vector of y -values, and a vector of z -values. Successive (x, y, z) triples from these vectors are graphed in 3-space:

```
>> plot3( xvec, yvec, zvec );
```


For example, if `xsol` is a solution variable from `ode45` when solving a system of three equations, we can plot the trajectory in the three-dimensional phase space with

```
>> plot3( xsol(:,1), xsol(:,2), xsol(:,3) );
```

if `x0` is the initial condition used in the solution, we could add that by using `plot3` to plot the point:

```
>> hold on;
```

```
>> plot3( [x0(1)], [x0(2)], [x0(3)], 'r.', 'MarkerSize', 20 );
```

Three-dimensional graphics figures have a rotate-tool button () on the tool bar; clicking that will allow clicking and dragging the graph to rotate the image. As you do this, in the bottom left corner of the graph that *azimuth* and *elevation* of the viewpoint are shown. If you get an orientation you like (e.g., 120,20), you can set that in your *MATLAB* code with

```
>> view([120,20]);
```

(This is sort of like the *axis* command for 2D plots: it sets the azimuth (angle from the $-y$ axis) and elevation (angle above the xy -plane) of the viewer.)

3. BACKGROUND

In this lab we consider the Lorenz equations,

$$(1) \quad \begin{aligned} x' &= \sigma(-x + y) \\ y' &= rx - y - xz \\ z' &= -bz + xy, \end{aligned}$$

a three-dimensional system with applications to weather modeling. These equations model the motion of a layer of fluid when the temperatures at the top and bottom boundaries of the layer differ. The variables are x , a measure of the intensity of the motion of the particles in the fluid; y , measuring the temperature difference between ascending and descending particles; and z , a measure of the distortion from vertical in particles' motion. The coefficients σ , b , and r are all positive, and represent different characteristics of the system: in particular, r is proportional to the difference in temperature between the boundaries of the layer. (The other parameters, σ and b , depend on the gas and geometry of the layer.) In this lab, we study how the behaviors of solution trajectories change for different values of r , and will see chaotic behavior and a new (to us) type of bifurcation called "period doubling."

4. LAB REPORT


In your lab report, you will write a paper that looks at the mathematics of the models developed in the Pre-Lab, and how they describe the functioning of the laser. In particular, you will examine

- How your linear analysis of the system at the different critical points allows you to predict its behavior when $r < 24.7368\dots$, and how this is different when $r > 24.7368\dots$
- How the case $r > 24.7368\dots$ exhibits sensitivity to initial conditions. Use your work from Part B, Exercise 1 to demonstrate on this, and reflect on what it means for weather forecasting.
- What we mean by chaotic behavior and how the Lorenz system exhibits this, along with the implications of this for weather forecasting.

5. PART A EXERCISES

For all exercises, we consider the Lorenz system, (1), with $\sigma = 10$ and $b = \frac{8}{3}$. Values of r , and consideration of the linearization of the system, are indicated in the exercises.

Exercise 1. Review with your partner your work in Exercise 2 of the prelab, determining the linear stability of the critical point $(0, 0, 0)$. Suppose we solve (1) with $r = 0.5$ and the initial condition $(x(0), y(0), z(0)) = (0.5, 0.5, 0.5)$. Based on your linear analysis, what do you expect the component plots of solutions to the system to look like? As $t \rightarrow \infty$, what values do you expect x , y , and z to approach?

Solve (1) numerically using `ode45` and plot x , y and z as functions of t , with the initial condition $(x(0), y(0), z(0)) = (0.5, 0.5, 0.5)$ to confirm that they do what you expected. For systems of two equations we also considered plots in the phase plane: graphs of y vs x . Here, because there are three state variables, we have a phase space instead of a phase plane. Use `plot3` (see the *MATLAB* section, above) to graph the trajectory you obtained in the phase space. Note that you can rotate the 3D figure by clicking the rotate-tool button  and then clicking and dragging the graph.

Then solve the system with the initial condition $(x(0), y(0), z(0)) = (0, 0, 0)$. How is the result different? What is happening in this case?

Exercise 2. Next, suppose we increase r to $r = 1.25$. Recall what your work in Exercises 2 and 4 of the prelab says about the stability of the different critical points of the system in this case. If we solve the system (1) now, with the initial condition $(x(0), y(0), z(0)) = (0.5, 0.5, 0.5)$, what do you expect the component plots of solutions to the system to look like? As $t \rightarrow \infty$, what values do you expect x , y , and z to approach?

Solve (1) numerically in this case using `ode45` and plot x , y and z as functions of t , with the initial condition $(x(0), y(0), z(0)) = (0.5, 0.5, 0.5)$ to confirm that they do what you expected. Then use `plot3` to graph the trajectory you obtained in the phase space.

Then solve the system with the initial condition $(x(0), y(0), z(0)) = (0, 0, 0)$. How is the result different? What is happening in this case?

Exercise 3. Finally, suppose we consider $r = 23$ and $r = 25$. What does your linear analysis suggest about trajectories in these cases? Use `ode45` to generate two more numerical solutions to (1) with these values of r . Plot the trajectories in phase space (you'll want to do this in two different figures), and the component plot of $x(t)$ for each.

When you have completed these exercises, think about what they tell you about the insight your linear analysis gives you on the system, and its nonlinear behavior. You want to relate four pieces of information: (1) the eigenvalues of the Jacobian at the different critical points; (2) the linear stability of the critical

points; (3) the long term behavior of the state variables x , y , and z ; and (4) the appearance of the trajectories in the phase space.

REFERENCES

- [BB] Brannan, James R, and William E Boyce. *Differential Equations: an Introduction to Modern Methods And Applications*. Third edition. Hoboken, NJ: Wiley, 2015.
- [CS] Sparrow, Colin. *The Lorenz Equations: Bifurcations, Chaos, and Strange Attractors*. New York, NY: Springer-Verlag, 1982.