LAB 5: THE LORENZ SYSTEM AND WEATHER PATTERNS, PART B

1. Matlab

MATLAB commands we use in this lab include the following.

1.1. ode45.

Finds a numerical approximation to a differential equation or system of equations:

```
>> [tsol,xsol] = ode45(f_handle, [tmin tmax], init_cond);
```

We can set options that change the behavior of ode45 with odeset; e.g., to set the maximum allowed error to 1×10^{-8}, we define an options object with:

```
>> options = odeset( 'RelTol', 1e-8 );
```

and then include this as an argument to ode45:

```
>> [tsol, xsol] = ode45(..., options);
```

1.2. plot.

Plot one vector against another; e.g.,

```
>> plot( tsol, xsol(:,1) );
```

1.3. plot3.

Plot a three-dimensional figure; input are a vector of x-values, a vector of y-values, and a vector of z-values. Successive (x, y, z) triples from these vectors are graphed in 3-space:

```
>> plot3( xvec, yvec, zvec );
```

e.g., given the ode45 command above, we can plot the trajectory for a system of three equations in the three-dimensional phase space with

```
>> plot3( xsol(:,1), xsol(:,2), xsol(:,3) );
```

if `x0` is the initial condition used in the solution, we could add that by using `plot3` to plot the point:

```
>> hold on;
>> plot3( [x0(1)], [x0(2)], [x0(3)], '.', 'MarkerSize', 20 );
```

We rotate graphics using the rotate-tool button () on the tool bar: clicking that will allow clicking and dragging the graph to rotate the image. If we have a desired *azimuth* and *elevation*, we can set these explicitly with

```
>> view([120,20]);
```

1.4. PlotComet_3D.

This is not a native *MATLAB* command; it is an add-on command available from *MathWorks*, the makers of *MATLAB*. Download it from the course web page. It plots an animated 3D plot of the trajectory specified by three vectors of x, y, and z values. For example, given the solution `xsol`, above, we could plot the trajectory (with a "comet" tail) with

```
>> PlotComet_3D( xsol(:,1), xsol(:,2), xsol(:,3) );
```

There are two options for `PlotComet_3D` that are very useful; `Frequency` dictates the speed with which the plot advances, and `blockSize` determines the length of the tail. Thus, we could speed up the display and pick a tail length with

```
>> PlotComet_3D( xsol(:,1), xsol(:,2), xsol(:,3),...
        'Frequency', 100, 'blockSize', 100 );
```

2. Background

In this lab we consider the Lorenz equations,

$$
\begin{aligned}
x' &= \sigma(-x+y) \\
y' &= r\,x - y - xz \\
z' &= -b\,z + xy,
\end{aligned}
\tag{1}
$$

a three-dimensional system with applications to weather modeling. We saw in the Prelab and Part A of the lab that possible critical points are $(0,0,0)$ and $P_\pm = (\pm\eta, \pm\eta, r-1)$ (where $\eta = \sqrt{b(r-1)} = \sqrt{\frac{8}{3}(r-1)}$). We can linearize (1) about these critical points to determine the qualitative behavior of the system.

At the end of Part A, we saw that when r is sufficiently large (in fact, when $r > r_{c2} = 24.73684211$) all three of the critical points are unstable, which suggests that the variables in the system either diverge (go to ∞) or are constrained by some sort of nonlinear feedback. We saw the latter behavior with the van der Pol equation in lab 2, when the critical point became unstable and a *limit cycle*—closed periodic trajectory in the phase plane—became stable.

For the Lorenz equations, it turns out that there is a similar behavior: trajectories constrained to an attractor (like the limit cycle in the van der Pol equation) are possible. In this case, however, we can see a "period doubling bifurcation" in which periodic solutions see their periods double as a parameter (r) is changed (decreased), and chaotic behavior is possible. We do not have the time to define or explore chaos here, but a reasonable summary is that trajectories are unpredictable but constrained to a specific region of phase space. In any case, the key points we want to remember are that (1) the system's behavior near critical points is well-approximated by the linearizations of the system, and (2) the nonlinear system may have significantly different behavior away from the critical points.

3. Part B

Before starting on these exercises, review your work from Part A. In particular, make sure that you understand what you expect trajectories in the phase space to do if $r < 1$ (Exercise 1), $1 < r < 1.3$ (Exercise 2), and $1.3 < r < 24$ (Exercise 3). When $r > 24.7368$, what happens to the stability of the critical points?

Exercise 1. Look at a value of r a bit bigger than $r = 25$ (e.g., $r = 28$). Numerically solve (1) with ode45 and plot the component x as a function of t, as well as the trajectory in the phase space. Be sure you can find where the critical points are in both graphs. Does the trajectory diverge to infinity? The set of points that the trajectory is able to reach in the phase space is called a *strange attractor*—it is a non-periodic, unpredictable, attracting solution.

Whenever we see trajectories with a great deal of variation, like this, we should be cautious about trusting the numerical result we obtain to be quantitatively accurate. Try setting the required tolerance for the ode45 solver to a much higher value, e.g., with

```
>> options = odeset( 'RelTol', 1e-8 );
```

(to require a relative error less than or equal to 10^{-8}), and then solve the system again:

```
>> [t1a, x1a] = ode45(..., options);
```

Is the result different? What does this suggest about your ability to predict values of the state variables for large times?

Exercise 2. To get a sense of what the trajectory in Exercise 1 is actually doing, it's useful to plot it in phase space as a parametric curve parameterized by time. The PlotComet_3D command will do this for you. Try running it to see what is happening to the trajectory. You will probably want to play with the Frequency and blockSize options to get a graph that gives a sense of what is happening.

There are a number of parameter ranges in which the Lorenz system goes through a period doubling cascade. It isn't obvious where these will occur, nor when the behavior will converge to a limit cycle instead of being chaotic—this is one of the things that makes nonlinear systems very interesting! Note that the Lorenz system (1) doesn't even look particularly messy—the nonlinearity is "just" quadratic. Yet its behavior can be very unintuitive. It was proposed as a simple meteorological model, but any "real" model is much more complicated. One might reasonably guess that one of the ramifications of that added complexity will be an increased difficulty in accurate long-term analysis.

Exercise 3. Finally, let's look at the behavior for some other values of r. Consider $r = 170$. Generate a phase space trajectory for this case. Is it significantly different from the behavior you saw in Exercises 1 and 2? (To check this, you may need to look for large enough times that any transient has vanished.)

Then, what happens if you consider a smaller value of r (say, $r = 160$)? Try decreasing r below $r = 150$—what happens to the trajectories in this case? You should see the number of loops in the limiting trajectory suddenly become finite. Continue decreasing r, considering between $r = 149$ and $r = 147$; what happens to the number of loops in the limiting trajectory? If we think about r *decreasing* rather than increasing, we call this a *period doubling cascade*; make sure that this description makes sense. What happens when you get to a still smaller value of r (e.g., $r = 145$)?

4. Lab Report

Review the background description of the Lorenz system as a model of the motion of fluid between two layers, especially in the Prelab and Part A. Note that the functions $x(t)$, $y(t)$, and $z(t)$ don't model the motion of individual particles. Instead, they describe the intensity of the motion of the particles in the fluid (x), the temperature difference between ascending and descending particles (y), and distortion from vertical motion of the particles (z). Then consider the lab report as described below.

Next, we posit that you have had the revelation that the unifying theme in your varied lab writing career is your overwhelming love of mathematical modeling, and so have founded a consulting firm specializing in modeling and the analysis of mathematical models. A popular science reporter has contacted you to consider the impact of climate change, which has the effect of increasing the temperature at the Earth's surface, on weather forecasting. You are writing a report in response to her request using a simple model (the Lorenz system) that captures some of the behavior of the atmosphere while avoiding the need to explain a far more complex model.

In your report you will want to address the questions

- How your linear analysis of the system at the different critical points allows you to predict its behavior when $r < 24.7368\ldots$, and how this is different when $r > 24.7368\ldots$
- How the case $r > 24.7368\ldots$ exhibits sensivity to initial conditions. Use your work from Part B, Exercise 1 to demonstrate on this, and reflect on what it means for weather forecasting.
- What we mean by chaotic behavior and how the Lorenz system exhibits this, along with the implications of this for weather forecasting.

Your lab report should have the following format:

I. **Introduction:** Summarize the purpose and contents of your report. You should include the system (1), briefly noting its relation to weather systems, but otherwise should keep technical notation to a minimum.

II. **Body:** In the body of the report, you should address the points noted above. You will want to include relevant equations, calculations, and graphs. Note that in doing this you should explain how your work in math 216 allows you to analyze this system. In your discussion you should also highlight the bifurcations that occur in the Lorenz model, how the behavior of solutions may change as a parameter (r) changes, and how the behavior for larger values of r may speak to our ability long-term weather forecasting.

III. **Conclusion:** Provide a short, several paragraph, summary of your results that ties together the work you have described in the body.

References

[BB] Brannan, James R, and William E Boyce. *Differential Equat ions: an Introduction to Modern Methods And Applications.* Third edition. Hoboke n, NJ: Wiley, 2015.

[CS] Sparrow, Colin. *The Lorenz Equations: Bifurcations, Chaos, and Strange Attractors*. New York, NY: Springer-Verlag, 1982.