LAB 5: THE LORENZ SYSTEM AND WEATHER PATTERNS

1. Objectives and Instructions

1.1. **Model.** The Lorenz equations

$$\begin{aligned} x' &= \sigma(-x+y) \\ y' &= r\,x - y - xz \\ z' &= -b\,z + xy \end{aligned} \tag{1}$$

are a nonlinear three-dimensional system which models the motion of a layer of fluid when the temperatures at the top and bottom boundaries of the layer differ. In this system, x measures the intensity of the motion of the particles in the fluid, y measures the temperature difference between ascending and descending particles, and z is a measure of the distortion from vertical in particles' motion. Gases are considered fluids in this context, so this system has applications to meteorological problems in which the "fluid" is taken to be Earth's atmosphere. The coefficients σ, b, and r are all positive, and represent different characteristics of the system: in particular, r is proportional to the difference in temperature between the boundaries of the layer. (The other parameters, σ and b, depend on the gas and geometry of the layer.) In this lab, we will study how the behaviors of solution trajectories for the Lorenz equations change as we vary r. As we do this we will see that the system undergoes several bifurcations,[1] and may exhibit interesting nonlinear behavior including chaos and period doubling.[2]

1.2. **Objectives.** Our goals in this lab are to see how we can linearize nonlinear systems such as (1), and to see some of the interesting nonlinear behaviors that may arise in such systems. In particular, we want to explore

- how Jacobians may be used to linearize nonlinear systems at a critical point,
- how the behavior of the linear system near critical points can help us understand the behavior of the nonlinear system,

[1] What is a bifurcation? Bifurcations can occur in systems that have a parameter—e.g., r—that determines the critical points. We say there is a bifurcation at a value of the parameter where the number of critical points changes (e.g., if a critical point is at $x=\sqrt{r}$, then as r goes from negative to positive values we go from zero to one critical point). We call those parameter values (e.g., $r=0$) bifurcation points. One more thing: sometimes we'll say there is a bifurcation occurs when the number of critical points stays the same, but their stability changes.

[2] We do not formally define chaotic behavior here, but a reasonable summary is that chaotic trajectories are unpredictable (we can't predict where they will be in phase space at any given time) but are constrained to a specific region of phase space. Period doubling occurs when periodic solutions see their periods double as a parameter (here, r) changes.

- and see how system (1) can display chaotic behavior, with solutions that diverge unpredictably from initial conditions, and period doubling.

2. Pre-Lab

Here, by "isolated," we mean what you think: there isn't another critical point in some circular region around the critical point. We look at this (again!) in §7.2, and will see there another explanation for why the Jacobian provides the coefficient matrix for the linearized system.

Earlier in the course, we saw how we can linearize nonlinear equations using Taylor series, and addressed linearization of nonlinear systems at critical points other than the origin by changing coordinates and then discarding small nonlinear terms. In the following we develop a more systematic approach to this process. Let $\mathbf{x}_0 = (x_0, y_0)$ be an isolated critical point of the autonomous system

$$
\begin{aligned}
x' &= f(x,y) \\
y' &= g(x,y).
\end{aligned}
\tag{2}
$$

Partial derivatives show up in multivariable calculus! If you have already taken such a course, this may look easy; if you haven't, it's easy to see how it works.

(Note that (1) is of this form—with three variables.) In matrix form, we write $\mathbf{x}' = \mathbf{f}(\mathbf{x})$, where $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\mathbf{f}(\mathbf{x}) = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix}$. The functions $f(x,y)$ and $g(x,y)$ have two independent variables x and y, and we can take partial derivatives of f or g with respect to either one.

Example 1: *Let $f(x,y) = x + y^2 + x^2 e^{xy}$. Find the partial derivatives of f with respect to x and y.*

Solution: The partial derivative of f with respect to x is calculated by taking the derivative of f using x as the variable, and treating y as a constant:

$$\frac{\partial f}{\partial x} = 1 + 2xe^{xy} + x^2 y e^{xy}.$$

Likewise the partial derivative with respect to y is calculated by treating x as a constant:

$$\frac{\partial f}{\partial y} = 2y + x^3 e^{xy}.$$

We use the notation $f_x = \frac{\partial f}{\partial x}$ and $f_y = \frac{\partial f}{\partial y}$ to more concisely denote the partial derivatives.

The *Jacobian* of (2) at the critical point $\mathbf{x}_0$ is the matrix

$$\mathbf{J}(\mathbf{x}_0) = \begin{pmatrix} f_x(\mathbf{x}_0) & f_y(\mathbf{x}_0) \\ g_x(\mathbf{x}_0) & g_y(\mathbf{x}_0) \end{pmatrix}.$$

We use the Jacobian to give a linear approximation to a vector function $\mathbf{f}(\mathbf{x})$, where $\mathbf{x} = (x(t), y(t))$, near $\mathbf{x}_0 = (x_0, y_0)$:

$$\mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{x}_0) + \mathbf{J}(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0) + \mathbf{e}(\mathbf{x}). \tag{3}$$

Note that thes expression in (3) looks very much like a the linear approximation for a function of one variable: $f(x) \approx f(x_0) + f'(x_0)(x - x_0)$!

Here we have defined $\mathbf{e}(\mathbf{x}) = \mathbf{f}(\mathbf{x}) - \mathbf{J}(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0)$.[3] Next, if $\mathbf{f}$ is the right-hand side of (2) and $\mathbf{x}_0$ is a critical point we know $\mathbf{f}(\mathbf{x}_0)$ must be zero, and we get

$$\mathbf{f}(\mathbf{x}) = \mathbf{J}(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0) + \mathbf{e}(\mathbf{x}). \tag{4}$$

[3] So, it is the error in the linear approximation.

If we assume that $\mathbf{e}(\mathbf{x})$ is small, we have the linear approximation of $\mathbf{f}(\mathbf{x})$. Let's do this for (2): if the critical point $\mathbf{x}_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$, take $u = x - x_0$ and $v = y - y_0$. Then $\mathbf{x}' = \begin{pmatrix} u' \\ v' \end{pmatrix}$, and $\mathbf{f}(\mathbf{x}) = \mathbf{J}(\mathbf{x}_0) \begin{pmatrix} u \\ v \end{pmatrix} + \mathbf{e}(\mathbf{x})$. Thus, if $\mathbf{e}(\mathbf{x})$ is small enough that we may ignore it, we obtain the linearization of (2) at the critical point $\mathbf{x}_0$,

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} f_x(x_0, y_0) & f_y(x_0, y_0) \\ g_x(x_0, y_0) & g_y(x_0, y_0) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f_x(x_0, y_0) \cdot u + f_y(x_0, y_0) \cdot v \\ g_x(x_0, y_0) \cdot u + g_y(x_0, y_0) \cdot v \end{pmatrix},$$

or, in matrix form, $\mathbf{u}' = \mathbf{J}(\mathbf{x}_0)\,\mathbf{u}$, where $\mathbf{u} = \begin{pmatrix} u \\ v \end{pmatrix}$.

Exercise 1: Recall the van der Pol system

$$\begin{aligned} x' &= y \\ y' &= -x + \mu(1 - x^2)y \end{aligned}$$

from Lab 2 has a single critical point at $(0, 0)$. For this system, we have $f(x, y) = y$ and $g(x, y) = -x + \mu(1 - x^2)y$. Find the Jacobian and use it to show that the linearization of the system near $(0, 0)$ is

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & \mu \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

as we found in Lab 2.

Analogously, we can define the Jacobian for 3×3 systems. Given the system

$$\begin{aligned} x' &= f(x, y, z) \\ y' &= g(x, y, z) \\ z' &= h(x, y, z), \end{aligned}$$

the Jacobian is $\mathbf{J} = \begin{pmatrix} f_x & f_y & f_z \\ g_x & g_y & g_z \\ h_x & h_y & h_z \end{pmatrix}$, and the linearization at (x_0, y_0, z_0) is

$$\begin{pmatrix} u' \\ v' \\ w' \end{pmatrix} = \mathbf{J}(x_0, y_0, z_0) \begin{pmatrix} u \\ v \\ w \end{pmatrix},$$

or $\mathbf{u}' = \mathbf{J}(\mathbf{x}_0)\,\mathbf{u}$. Note how this illustrates the power of our matrix notation. We can generalize from two to three (or more!) dimensions completely transparently!

2.1. **The Lorenz System.** In meteorological applications of the Lorenz system (1), the constants $\sigma = 10$ and $b = 8/3$ are fixed, but the parameter r can vary, as it is proportional to the temperature differences at the boundaries of the fluid layer. We want to understand how and when changes in the behavior of solutions to the system occur with changes in r.

Your $\mathbf{J}(0, 0, 0)$ in Exercise 2 should look like $\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & e \end{pmatrix}$. The characteristic polynomial for such a matrix is

$$(e - \lambda) \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix}.$$

Exercise 2: Write down the Jacobian for (1) and evaluate it at the critical point $(x_0, y_0, z_0) = (0, 0, 0)$. Calculate its eigenvalues (remember we consider only $r > 0$) and determine how the stability of the origin changes with r. (You do not need to determine the type of the critical point, only its stability for r in various intervals.)

2.2. **Bifurcations.** A common theme in our labs has been the question of how solutions to differential equations change as we vary some parameter in the model. The value of the parameter at which the solutions change dramatically is called a bifurcation point. For example, critical points (and other attracting sets, such as limit cycles) may appear or disappear, and the stability of the critical points (and other attracting sets) may change.

Example 2*: We have seen several types of behaviors that can occur as some parameter in our differential equation (or system) passes a bifurcation point.*

(1) *In Exercises 2.5.10–2.5.12 in* [BB] *we see examples of saddle-node, pitchfork, and transcritical bifurcations. Find the critical points for each, noting how the number and stability of critical points changes at the bifurcation points.*
(2) *Review Lab 2 to see how the stability of the origin as a critical point changes as we vary μ in the van der Pol equation. In this case a bifurcation diagram would be three-dimensional, so we don't try to draw it. Instead, indicate how the stability of the critical point (the origin) changes as μ changes. What other change occurs in the system? This bifurcation is called a (supercritical) Hopf bifurcation.*

Solution:

(1) For Exercise 2.5.10, we consider the differential equation $x' = f(x) = r - x^2$. Critical points are when $x' = 0 = r - x^2$. Thus if $r < 0$ there are no critical points, if $r = 0$ there is the unique critical point $x = 0$, and if $r > 0$ there are the two critical points $x = \pm\sqrt{r}$. Thus as r goes from negative values to positive ones, we see a transition from no solutions to one solution to two. Note that $f'(x) = -2x$, so $x = 0$ is semistable, $x = \sqrt{r}$ is stable,[4] and $x = -\sqrt{r}$ is unstable.
For Exercise 2.5.11, we consider the equation $x' = f(x) = x(r - x^2)$. Here $x = 0$ is always a solution, and for $r > 0$ we have also the critical points $x = \pm\sqrt{r}$. Noting that $f'(x) = r - 3x^2$, if $x = 0$ and $r < 0$, we have $f' < 0$ and the critical point is stable. If $r > 0$, we see similarly that $f'(\pm\sqrt{r}) < 0$ and $f'(0) > 0$, so that $x = 0$ is unstable and $x = \pm\sqrt{r}$ are stable.

[4]Recall that in §1.2 of [BB] we saw that the linearization of $y' = f(y)$ tells us that if for a critical point y_0 we see $f'(y_0) < 0$, the critical point is asymptotically stable, with the expected extension to the cases $f'(y_0) > 0$ or $f'(y_0) = 0$.

For Exercise 2.5.12, we consider $x' = f(x) = x(r-x)$, and solutions are $x = 0$ and $x = r$. Then $f'(x) = r - 2x$, and if $r < 0$ we see that the point $x = 0$ is stable and $x = r$ unstable; and for $r > 0$, the opposite is true.

(2) In Lab 2, we saw that the critical point of the van der Pol system (see Exercise 2) is $(0, 0)$, and that the eigenvalues of the linearized system are $\lambda = \mu \pm i$, which have negative real part when $\mu < 0$ and positive real part when $\mu > 0$. Therefore, the critical point is stable when $\mu < 0$ and unstable when $\mu > 0$. When $\mu > 0$, we saw that trajectories spiral away from the origin, but are bounded by a periodic trajectory. This is a limit cycle (we do not prove this here, but it can be shown that the limiting trajectory is a periodic solution to the nonlinear equation). Thus as μ passes from negative to positive values, the origin becomes unstable, but a new type of attracting set (the limit cycle) appears.

Exercise 3: For the Lorenz system (1), find all critical points in terms of the parameter r. For what values of r is there a single (real) critical point? More than one? What is the bifurcation point (that is, the value of r where the number of critical points changes)?

Exercise 4: Let $P_+ = (\eta, \eta, r-1)$, where $\eta = \sqrt{\frac{8}{3}(r-1)}$. Find the Jacobian $\mathbf{J}(P_+)$. The characteristic polynomial of this is $p(\lambda) = \det(\mathbf{J}(P_+) - \lambda\mathbf{I}) = -\lambda^3 - \frac{41}{3}\lambda^2 - \frac{8}{3}(r+10)\lambda + \frac{160}{3}(1-r)$. Graph the characteristic polynomial for four choices of r: one in the interval $(1, 1.34562)$, one with $r = 1.34562$, one in the interval $(1.34562, 24.7368)$, and one with $r > 24.7368$. Use your graphs to determine the type and stability of the critical point in the first three cases. Is it obvious what is happening when $r > 24.7368$? (The values $r = 1.34562$ and $r = 24.7368$ are those where significant changes in the behavior of the system occur.) Describe what properties of the critical point you can see changing as r crosses these values.

References

[BB] Brannan, James R, and William E Boyce. *Differential Equations: an Introduction to Modern Methods And Applications.* Third edition. Hoboken, NJ: Wiley, 2015.

[CS] Sparrow, Colin. *The Lorenz Equations: Bifurcations, Chaos, and Strange Attractors*. New York, NY: Springer-Verlag, 1982.