Math 425–4, Fall 1997
Prof. Hochster

Extra Credit Problems VI

EC#18. Assume that, on average, 4 children are born every second (the actual number: 4.083). Treat the number of births in a time interval as a Poisson random variable (not quite accurate unless population is stable). What is the probability that there will be no births in a specific minute, e.g., from 1 p.m. to 1:01 p.m. on October 31, 1997? [10/15]

EC#19. With the same assumptions as in **EC#18.**, what is the probability that during the year 1998 there will be at least one minute, starting at an integer number of minutes after the hour, during which there are no births? [10/15]

EC#20. Show that the answer to **EC#19.** is quite small even if there is no restriction on when the minute starts (e.g., it might start at 4.17869544 minutes after 2 a.m. or at any other time), and one considers a period of 100 billion years. [No due date yet.]

EC#21–23. Here is one way of computing all of the moments of a binomial random variable X with parameters n and p. Let X_i denote a random variable that has the value 1 if there is a success on the i th trial and 0 otherwise, so that $X = X_1 + \cdots + X_n$. The steps below give a formula for the r th moment of X, $E[X^r]$. Each of (a) & (b), (c) & (d), and (e) & (f) counts as one problem.

(a) First note that X^r is the sum of all the terms $\binom{r}{a_1\ \cdots\ a_n} X_1^{a_1} \cdots X_n^{a_n}$ where $a_1, \ldots, a_n$ runs through all n-tuples of nonnegative integers that add up to r, by the multinomial theorem. Conclude that $E[X^r]$ is the sum of all the terms $\binom{r}{a_1\ \cdots\ a_n} E[X_1^{a_1} \cdots X_n^{a_n}]$, where $a_1, \ldots, a_n$ varies as above. Explain why, very briefly.

(b) Let t denote the number of variables X_i that occur with a strictly positive exponent in $X_1^{a_1} \cdots X_n^{a_n}$. Explain why $E[X_1^{a_1} \cdots X_n^{a_n}] = p^t$.

(c) Note that the values of t that occur are between 1 and the minimum of the integers r and n. Show that $E[X^r]$ can be expressed in the form $c_r p^r + c_{r-1} p^{r-1} + \cdots + c_2 p^2 + c_1 p$, where c_t is the sum of all the multinomial coefficients $\binom{r}{a_1\ \cdots\ a_n}$ such that precisely t of the a_i are positive (and the others are zero). Note: $c_t = 0$ if $t > n$ (or $t > r$).

(d) Show that c_t is the number of words of length r that can be formed from n letters (order matters, and repetitions are allowed) such that exactly t letters occur. Hence, $c_t = \binom{n}{t} Q(t, r)$ where $Q(t, r)$ denotes the number of words of length r that can be formed from t letters such that every letter occurs at least once — explain why.

(e) Prove (using inclusion-exclusion for counting finite sets) that

$$Q(t, r) = t^r - \binom{t}{1}(t-1)^r + \binom{t}{2}(t-2)^r - \cdots + (-1)^{t-1}\binom{t}{t-1}1^r \qquad \text{and so}$$

$$E[X^r] = \sum_{t=1}^{r} \left(\binom{n}{t} \sum_{j=0}^{t-1} \left((-1)^j \binom{t}{j} (t-j)^r \right) \right) p^t.$$

(f) Evaluate $E[X]$, $E[X^2]$, and $E[X^3]$ using this formula.

[10/17]