Math 425–4, Fall 1997
Prof. Hochster

Extra Credit Problems VIII

EC#27. A cookie monster has two jars of cookies, a larger jar that initially has $2N$ cookies in it, and a smaller one that initially has N cookies in it, where N is a positive integer. Each time the monster wants a cookie, it throws an ordinary die, and if the number that comes up is one or two, it eats a cookie from the smaller jar, while if the number that comes up is 3, 4, 5, or 6, it eats a cookie from the larger jar. Let p_N be the probability that the smaller jar runs out of cookies first. Find a formula for p_N (can be as a sum) and evaluate it for several values of N. Make the best conjecture that you can about the behavior of p_N. For triple credit, prove your conjecture. [No deadline on this part.] (This is somewhat related to the Banach match problem discussed in Example 9c. on p. 165 of the text, but is quite different in several respects and is **not** known as the Banach cookie problem.) [11/7]

EC#28. Consider an event like occurrence of earthquakes such that the number of occurrences in a time interval of length 1 is given by a Poisson random variable with parameter λ, so that the number of occurrences in a time interval of length t is given by a Poisson random variable with parameter λt. Consider two disjoint time intervals of length s and t. Let Y be the total number of occurrences in the two given disjoint time intervals and let Z be the number in a time interval of length $s+t$. Compare the mass functions for Y and Z. [11/7]

EC#29. Let X be a continuous random variable with mean μ and variance V. Let f be the density function for X. Let c be a constant. Define $g(x) = f(x-c)$. Show that g is also the density function of a random variable Y, and compute the mean and variance of Y in terms of μ, V and c. [Hint: consider $X+c$.] [11/10]

EC#30. Let X be a continuous random variable with mean μ and variance V, and let s be a positive constant. Let f be the density function for X, and let $g(x) = (1/s)f(x/s)$. Show that g is also the density function of a random variable Z, and compute the mean and variance of Z in terms of μ, v and s. [Hint: consider sX.] [11/10]

EC#31. Show that $\Phi(x)$ is given by the power series expansion

$$\frac{1}{2} + \frac{1}{\sqrt{2\pi}}\Big(x - \frac{x^3}{2^1\,1!\,3} + \frac{x^5}{2^2\,2!\,5} - \frac{x^7}{2^3\,3!\,7} + \frac{x^9}{2^4\,4!\,9} - \cdots + (-1)^k \frac{x^{2k+1}}{2^k\,k!\,(2k+1)} + \cdots\Big).$$

This series converges for all real x. Show that for $0 \le x \le 1$, the terms of this series up to the degree 7 term give a value correct to at least three decimal places. [$\Phi(x) = 1/2 + \frac{1}{\sqrt{2\pi}}\int_0^x e^{-y^2/2}\,dy$ (explain this). Subsituting $u = -y^2/2$ in the usual expansion for e^u gives a power series that can be integrated term by term.] [11/12]

EC#32. Consider repeated independent trials of an experiment whose numerical outcome is given by a continuous random variable, and let X_i denote the outcome of the i th trial, so that $X_1, X_2, X_3, \ldots$ are continuous random variables, are independent, and all have the same distribution. Let Y denote the discrete random variable whose value is smallest number n of trials at which one has $X_n > X_{n+1}$. Find $E[Y]$. [No deadline yet.]