Math 425–4, Fall 1997
Prof. Hochster

Comments on Beta Distribution

Examples where beta distributions come up: Given a Poisson process, the distribution of the fraction of the time until $r+s$ events occur at which the r th event occurs has beta distribution with parameters r, s. More generally, if X, Y are gamma random variables with parameters λ, r and λ, s, respectively, the distribution of $X/(X+Y)$ is beta with parameters r, s. The fraction of the time needed for a large project to accomplish a certain part of that project often has beta distribution. If you have n independent uniform random variables on $[0, 1]$, the distribution of the k th smallest is beta with parameters k, $n-k+1$. If you have a large lot of manufactured items and a sample, with replacement, of size $r+s$ which has exactly r defectives, then the distribution function for the fraction of defectives is beta with parameters $r+1$ and $s+1$.

Extra Credit Problems IX

EC#33. Prove that for all $b > 0$, $\Phi(b) > 1 - \frac{1}{\sqrt{2\pi} b e^{b^2/2}}$. This implies $\Phi(b) = 1.0000$ to four places for $b \geq 4$. (Note that $1 - \Phi(b) = \int_b^\infty e^{-x^2/2}\,dx$. Since $\frac{x}{b} \geq 1$ on the interval of integration, increasing the integrand to $\frac{x}{b}\, e^{-x^2/2}$ can only increase the value of the integral.) [11/12]

EC#34. Show directly that $\mathrm{Var}(X) = t/\lambda^2$, where X is a gamma random variable with parameters t and λ. The text omits this calculation. (Note: an argument was given in class for the special case where t is an integer, but it not valid in general.) [11/12]

EC#35. Show directly from the definition of the beta function as an integral (i.e., without using the formula in terms of the gamma function) that the following hold for all a, $b > 0$:

(a) $B(a, b) = B(b, a)$

(b) $B(a, b) = \frac{b-1}{a} B(a+1, b-1)$ if $b > 1$. (Integrate by parts, using that $x^{a-1} = d(\frac{1}{a}x^a)$.)

(c) $B(a, 1) = 1/a$.

(d) $B(a, b) = \dfrac{(a-1)!(b-1)!}{(a+b-1)!}$ when a and b are positive integers.

(e) $B(a, b+1) + B(a+1, b) = B(a, b)$ (expand $B(a, b+1)$ using the fact that $(1-x)^{b+1} = (1-x)(1-x)^b = (1-x)^b - x(1-x)^b$). [11/17]

EC#36. In this problem you are to fill in the details of an argument to show that $B(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$. First note that we can write $\Gamma(a) = \int_0^\infty e^{-x} x^{a-1}\,dx$ and $\Gamma(b) = \int_0^\infty e^{-y} y^{b-1}\,dy$ using distinct dummy variables x, y. Multiplying yields that $\Gamma(a)\Gamma(b) = \int_0^\infty \int_0^\infty e^{-(x+y)} x^{a-1} y^{b-1}\,dx\,dy$, reinterpreting the product of the integrals as an iterated integral. Now we want to evaluate this integral differently. It represents the volume between the surface $z = e^{-(x+y)} x^{a-1} y^{b-1}$ and the plane $z = 0$ lying over the first quadrant. Let $A(u)$ denote the area of the cross-section lying over the line segment $x + y = u$ in the first quadrant, connecting $(0, u)$ to $(u, 0)$. (Think of u as fixed, but later we shall let u vary.) The portion of the volume lying over the trapezoid bounded by the line segments corresponding to u and $u+\Delta u$ and the x and y axes is approximately the area of the cross-section, $A(u)$, multiplied by $\frac{1}{\sqrt{2}}\Delta u$, which is the height (or thickness) of the trapezoid. Thus, $\Gamma(a)\Gamma(b) = \frac{1}{\sqrt{2}}\int_0^\infty A(u)\,du$. We next want to find $A(u)$. Let t denote distance along the line segment $x+y=u$ measured from the endpoint on the y-axis. Then at the point (x,y) the value of t is $\sqrt{2}x$. Show that $A(u)$ is given by $e^{-u}\int_{t=0}^{t=\sqrt{2}u}(\frac{t}{\sqrt{2}})^{a-1}(u-\frac{t}{\sqrt{2}})^{b-1}\,dt$ (in this integration u is constant and t is varying, so that we can factor out e^{-u}). Let $s=\frac{t}{\sqrt{2}u}$. Show that this can be rewritten as $A(u) = \sqrt{2}e^{-u}\,u^{a+b-1}\int_{s=0}^{s=1}s^{a-1}(1-s)^{b-1}\,ds$, which is $\sqrt{2}e^{-u}\,u^{a+b-1}B(a,b)$. Now the formula for $\Gamma(a)\Gamma(b)$ to reach the conclusion that $\Gamma(a)\Gamma(b)=\Gamma(a+b)B(a,b)$, as desired. [11/17]

EC#37. In the game of gambler's ruin, two players start with i and $N-i$ dollars initially, and flip a coin (or participate in some other similar event): if the coin comes up heads, which has probability p, then the first player (who started with i dollars) wins a dollar from the second — otherwise, the second player wins a dollar from the first. They continue until one of the players has won all the money while the other has none. Let $q=1-p$. The problem is analyzed in the text (Example 4j., pp. 90–93 in Ross, Fifth Edition), where it is shown that the probability P_i that the first player wins all the money is $\dfrac{1-(q/p)^i}{1-(q/p)^N}$ if $p\neq\frac{1}{2}$ and is $\dfrac{i}{N}$ if $p=\frac{1}{2}$. You may assume this and also assume that, if Q_i is the probability that the second player wins all the money, then $P_i+Q_i=1$. Thus, while the game can go on forever, this has probability 0. Show that the formula for P_i is continuous, i.e., that $\displaystyle\lim_{p\to\frac{1}{2}}\frac{1-(q/p)^i}{1-(q/p)^N}=i/N$. (Here, i, N are fixed, but $q=1-p$.) [11/19]

EC#38. Consider a similar situation to that in **EC#37.** in which the the first player has i dollars, p and q have the same meaning as before, but the second player has unlimited resources. What is the probability that the first player eventually loses everything? [11/19]

EC#39. In the situation of **EC#37.**, what is the expected number E_i of turns before one of the players runs out of money, where one of them is starting with i dollars and the other with $N-i$? (One approach is to find a set of simultaneous equations satisfied by all the E_i for fixed N, and then solve them.) [11/21]

EC#40. Consider a game in which a player repeatedly rolls a fair die. If the die shows a 1 or 2 the player loses a dollar, if the die shows a 3 or 4 the player wins a dollar, and if the die shows a 5 or 6 the player wins 2 dollars. Let $P(i)$ be the probability that the player starts with i dollars but eventually loses everything.

(a) Show that $P(i+1)=P(i)P(1)$ for $i\geq 1$. Hence, $P(i)=P(1)^i$.

(b) Show that $P(1)<1$ by comparing this game to the version of gambler's ruin in **EC#38.** with $p=\frac{2}{3}$. (If the player won only 1 dollar instead of 2 with a roll of 5 or 6, this situation would be gambler's ruin.)

(c) By conditioning on the outcome of the first roll, find a cubic equation satisfied by $x=P(1)$, and find x (the correct equation has an integer root, which cannot be the correct answer, and factoring leads to a quadratic equation). [11/24]