Math 425–4 Final Examination Solutions
Prof. Hochster

Total points: 200.

1. [25] (a) [15] Six cards are chosen at random from an ordinary deck.

(1) What is the probability that they are all black? Ans.: $\binom{26}{6}/\binom{52}{6}$.

(2) What is the probability that three of the cards have the same rank and the other three also have the same rank (but necessarily different from the rank of the first three)? (E.g., 3 jacks and 3 sevens would be a success.) Ans.: (Choices for ranks)(choices for which 3 in each rank)/(all choices) = $\binom{13}{2}\binom{4}{3}\binom{4}{3}/\binom{52}{6}$.

(3) What is the probability that there are no aces among these cards? Ans.: $\binom{48}{6}/\binom{52}{6}$.

(b) [5] A password is to be constructed from the letters A, B, C, D, E. How many such passwords are there containing 10 letters if repetitions are allowed? (The order of the letters matters.) An example would be $BBDAAEBABA$. Ans.: 5^{10}.

(c) [5] In part (b), suppose that the password must contain 4 A's, 3 C's, and 3 D's, and that precisely 2 of the A's must occur among the first five letters (and 2 in the last five). How many such passwords are there?

(No. of ways to place first two A's)(no. of ways to place second two A's)(no. of ways after that to place the 3 C's) = $\binom{5}{2}\binom{5}{2}\binom{6}{3}$ (the places for the D's are determined, after those for the A's and C'S are chosen, although one might include $\binom{3}{3}$ as a factor).

2. [25] (a) [7] Find the conditional probability of throwing a 7 with a pair of fair dice (i.e, the 2 numbers showing add up to 7), given that sum of the dice is odd.

The dice can be first even, second odd in $3 \cdot 3$ ways, and the same for first odd, second even. Thus, the probability of having the sum be odd is $(9 + 9)/36 = 1/2$. The ways to get a seven are $1 + 6, 2 + 5, 3 + 4, 4 + 3, 5 + 2, 6 + 1$, so the probability of that is $6/36 = 1/6$. The answer is $(1/6)/(1/2) = 1/3$. Alternatively, one may think of it as (6 ways to get a seven)/(18 ways to get an odd total) $= 1/3$.

(b) In a certain state, sixty per cent of the homes have incomes at most \$35,000 and forty per cent have income greater than that. Thirty-five per cent of the lower income homes have a computer and eighty per cent of the higher income homes have a computer.

(1) [9] What is the probability that a randomly selected home from this state has a computer? Ans.: $(.6)(.35)+(.4)(.8) = .21 + .32 = .53$.

(2) [9] If a home is selected at random from this state and it turns out that it has a computer, what is the probability that the income for that home is at most \$35,000? $(.6)(.35)/(.53) = .21/.53 = 21/53$.

3. [25] The number of fires per week in a certain city is a Poisson random variable with parameter 3.

(a) [8] What is the probability that there will be no fires in a given week? Ans.: e^{-3}.

(b) [8] What is the probability that there will be 2 or more fires in a given week? Ans.: $1 - P(i = 0) - P(i = 1) = 1 - e^{-3} - e^{-3}3/1! = 1 - 4e^{-3}$.

(c) [9] What is the probability that there will be no fires in a certain week and precisely 1 fire in the following week? Explain briefly what you are using about Poisson random variables.

The probabilities for the first and second week considered separately are e^{-3} and $e^{-3}3$. Because the values of a Poisson random variable in disjoint time intervals are independent, the probability is the product, $e^{-3}e^{-3}3 = 3e^{-6}$.

4. [25] (a) [12] X is uniformly distributed on [0,2] and Y is an exponential random variable with density $f_Y(y) = 2e^{-2y}$ on $(0, \infty)$. Assume that X and Y are independent. Express $P(2 \le X + Y \le 3)$ as an iterated integral directly from the definitions. Sketch the region over which you are integrating. Do not evaluate.

X has density 1/2 (the interval has length 2), and so the joint density is the product, e^{-2y}. We need to integrate this over the region between the lines $x + y = 2$ and $x + y = 3$ on which the density is positve, which is bounded on the left by the y-axis and on the right by the line $x = 2$. This is $\int_0^2 \int_{2-x}^{3-x} e^{-2y}\, dy\, dx$.

(b) Suppose that the joint density function of X and Y is given by $f(x, y) = 2$ on the triangular region described by $x > 0$, $y > 0$, $x+y < 1$, which is a triangle bounded by the positive axes and part of the line $x + y = 1$, and suppose that the density is 0 outside this region. Carry through all parts of this problem completely.

(1) [4] Find the density of X, $f_X(x)$. Discuss the cases $x \le 0$, $0 < x < 1$ and $x \ge 1$.

This is $\int_{-\infty}^{\infty} f(x, y)\, dy$. If $x \le 0$ or if $x \ge 1$ the density vanishes and this is 0. If $0 < x < 1$, this is $\int_0^{1-x} 2\, dy = 2(1 - x) - 0 = 2 - 2x$.

(2) [1] By symmetry, what can you say about $f_Y(y)$?

It is 0 unless $0 < y < 1$ in which case it is $2 - 2y$.

(3) [4] What is the conditional density of X given $Y = y$, assuming that $0 < y < 1$?

This is $f(x, y)/f_Y(y) = 2/(2 - 2y) = 1/(1 - y)$ if $0 < x < 1 - y$, and vanishes otherwise, since then (x, y) is outside the region.

(4) [4] Find the conditional probability that $X < 1/5$, given that $Y = 3/4$.

Since $Y = 3/4$, X can vary only in $[0, 1/4]$, and has density $1/(1 - \frac{3}{4}) = 4$ in this interval. Thus, the probability is $(1/5)4 = 4/5$.

5. [25] The times of arrival of the first bus at a certain stop after the hour are uniformly distributed over a thirty minute interval, and independent. The average time of arrival X (in minutes after the hour) is computed for 25 buses. Estimate the probability that this average is 20 or more minutes. Make the steps of your thought process clear: what are you assuming, what distribution(s) are you using, and why, and so forth.

Let X_i be the arrival time of the ith bus. By the central limit theorem, X, the average of the X_i, is approximately normal. The mean will be 15 and the variance $30^2/12 = 75$ for any one of the X_i. The mean for X will therefore be 15, and the variance will be $25(75)/25^2 = 3$, so that $(X - 15)/\sqrt{3}$ will be unit normal. $P(X \ge 20) = P(\frac{X-15}{\sqrt{3}} \ge \frac{20-15}{\sqrt{3}}) = 1 - \Phi(\frac{5}{\sqrt{3}})$.

6. [25] (a) [15] X is a normal random variable with mean 8 and standard deviation 4. Y is a normal random variable with mean 7 and standard deviation 3. Assume that X and

Y are independent. Say as much as you can about $X+Y$. Is it normal? Are its mean and standard deviation determined, and, if so, what are they? Also, say as much as you can about $P(X+Y \geq 21)$.

Since X and Y are independent, $X+Y$ is normal with mean 8+7 and standard deviation $\sqrt{3^2+4^2}=5$. Thus, $P(X+Y \geq 21) = P(\frac{X+Y-15}{5} \geq \frac{21-15}{5}) = 1-\Phi(\frac{6}{5})$.

(b) [10] The cards from one deck are placed at random into 52 separarte envelopes. Forty of the cards from a second deck are placed at random into 40 of these envelopes (so that forty envelopes contain two cards and the rest have only one). What is the expected number of envelopes containing two identical cards (e.g., the seven of clubs from both decks)? If your answer is based on the use of indicator random variables, give a reasonable amount of explanation of your method.

Let X_i be 1 if the ith card from the second deck is placed in the envelope of a matching card and 0 if not, $1 \leq i \leq 40$. Let $X = X_1 + \cdots X_40$, which is the number we are interested in. Then $E[X] = E[X_1] + \cdots + E[X_{40}]$. Each $E[X_i]$ is $P(X_i = 1) = 1/52$, and so $E[X] = 40/52 = 10/13$.

7. [25] (a) [5] X and Y are random variables each with variance 2, and their covariance 3. What is the correlation? Ans.: $3/\sqrt{2 \cdot 2}$ (not really possible, since > 1)

(b) [15] X and Y are independent random variables. X has mean 4 and variance 12, while Y has mean 6 and variance 5. Find each of the following, if it is determined.

(1) [3] The mean of $X+Y$. Ans. $4+6=10$.

(2) [3] The mean of $X-Y$. Ans.: $4-6=-2$.

(3) [3] The variance of $X+Y$. Ans.: $5+12=17$.

(4) [3] The variance of $X-Y$. Ans. $5+12=17$.

(5) [3] The covariance of X and Y. Ans. 0 (they are independent).

(c) [5] X and Y are independent random variables. Their moment generating functions are given by $M_X(t) = e^{2t}$ and $M_Y(t) = 1+2t+t^2$. What is the moment generating function $M_{X+Y}(t)$ of $X+Y$? Ans.: Since they are independent, this is the product, $e^{2t}(1+2t+t^2)$.

8. [25] (a) [15] Let X be a random variable that is uniformly distributed on [0,1]. Let $Y = X^2$. Give a formula for the distribution function for Y, $F_Y(b)$. Treat the cases where $b \leq 0$, $0 < b < 1$ and $b \geq 1$. What is the density $f_Y(b)$ when $0 < b < 1$?

The distribution is 0 if $b \leq 0$ and 1 if $b \geq 1$. For $0 < b < 1$, $F_Y(b) = P(Y \leq b) = P(X^2 \leq b) = P(X \leq \sqrt{b}) = \sqrt{b}$. Differentiating, we find that the density is $(1/2)b^{-1/2}$ in $(0,1)$ (and 0 elsewhere).

(b) [10] Suppose that X and Y are independent random variables that are exponentially distributed. Suppose that X has mean 3 and that Y has mean 4. Set up an iterated integral for $E[X^3 - XY + Y^2]$. Indicate clearly the region over which the integral is to be taken. You need not calculate any integrals.

The densities are $\frac{1}{3}e^{-x/3}$ and $\frac{1}{4}e^{-y/4}$, for positive values of the variable and 0 otherwise. The joint density is therefore $\frac{1}{12}e^{-x/3}e^{-y/4}$ in the first quadrant and 0 elsewhere, and $E[X^3 - XY + Y^2] = \int_0^\infty \int_0^\infty (x^3 - xy + y^2)\frac{1}{12}e^{-x/3}e^{-y/4}\,dx\,dy$; the region is the first quadrant.