Math 425, Fall 1997

Prof. Hochster **Solutions to Problems from Assignment #11** (pp. 422–3)

4. (a) $20/15 = 4/3$ is a useless upper bound, since all probabilities are bounded above by 1. (If one uses that the values are integers one can replace 15 by 16, and one gets the similarly useless upper bound $20/16 = 5/4$.) (b) The sum X has mean 20 and variance 20, and $P(X > 15.5) = P(\frac{X-20}{\sqrt{20}} > \frac{15.5-20}{\sqrt{20}})$ which means that the probability $\approx .1 - \Phi(-4.5/\sqrt{20}) = \Phi(4.5/\sqrt{20}) = 1 - \Phi(1.006)) = .8438$. (I'll accept the use of 15 for 15.5 here.)

5. The individual errors have mean 0 and variance $1/12$. The sum X of the fifty errors is approximately normal with mean 0 and variance $50/12 = 25/6$. We want $P(X > 3) + P(X < -3) = 2P(X > 3) = 2P(\frac{X}{\sqrt{(25/6)}} > 3/\sqrt{(25/6)}) = 2(1 - \Phi(3/\sqrt{25/6}) = 2(1 - \Phi(3\sqrt{6}/5) = 2(1 - \Phi(1.47)) = 2(1 - .9292) = 2(.0708) = .1416$.

6. Let X_i be the outcome of the ith roll. This is $q = P(X_1 + \cdots X_{79} \le 300.5)$. Since each X_i has mean 3.5 and variance $(1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2)/6 - (3.5)^2 = (1 + 4 + 9 + 16 + 25 + 36)/6 - 49/4 = 35/12$, the sum has mean $79(3.5) = 276.5$ and variance $79(35/12) = 230.41667$ to five places and the standard deviation is the square root of this or 15.17948. Thus, $q \approx \Phi(\frac{300.5-276.5}{15.1798}) = \Phi(24/15.1798) = \Phi(1.58) = .9429$ (I'll accept the use of 300 instead of 300.5 here).

7. For one bulb the mean is 5 hours ($\lambda = 1/5$) and the variance is 25. Let X_i be the life of the ith bulb. Let $X = X_1 + \cdots + X_{100}$. X is approximately normal with mean 500 and variance 2500. We want $P(X \ge 525) = P(\frac{X-500}{50} \ge 25/50) = 1 - \Phi(.5) = 1 - .6915 = .3085$.

8. Let Y_i be the time it takes to replace the ith bulb, $1 \le i \le 99$. We want $P(X_1 + \cdots + X_{100} + Y_1 + \cdots + Y_{99} < 550)$. The sum of the X's is approximately normal with mean and variance as above. Each Y_i has mean .25 and variance $(.5)^2/12$, so the sum of the Y_i has mean $99(.25) = 24.75$ and variance $99/48 = 33/16 = 2.0625$. The sum of all the X's and Y's is approximately normal with mean 524.75 and variance $\sigma^2 = 2502.0625$, and $\sigma = 50.02062$ to five places. Thus, the answer is $\Phi(\frac{550-524.75}{50.02062}) = \Phi(.5048) = .6915$ if one rounds .5048 to .50. Interpolation (or computing Φ) yields a slightly better answer.

9. Think of X as the sum of n independent gamma random variables with parameters 1, 1. (These are exponential, so that each has mean $1/1$ and variance $(1/1)^2$.) It follows that X is approximately normal with mean n and variance n. Thus, X/n is approximately normal with mean 1 and variance $1/n$. Hence, $P(|X/n - 1| > .01) = P(X/n - 1 < -.01) + P(X/n - 1 > .01)$ We now divide by the standard deviation, $\sqrt{1/n}$, which is equivalent to multiplying by $\sqrt{n}$. This becomes $P(\frac{X/n-1}{\sqrt{1/n}} < -\sqrt{n}/100) + P(\frac{X/n-1}{\sqrt{1/n}} > \sqrt{n}/100)$. Since $\frac{X/n-1}{\sqrt{1/n}}$ is approximately standard normal for large n, the two terms are equal, and we get $\approx 2(1 - \Phi(\sqrt{n}/100))$ and we want this to be $\le .01$. This gives the condition $1 - \Phi(\sqrt{n}/100) \le .005$ or $\Phi(\sqrt{n}/100) \ge .995$, which, from the table, yields $\sqrt{n}/100 \ge 2.58$ or $\sqrt{n} \ge 258$, so we want $n \ge 258^2 = 66,564$.

10. Suppose there are n cars on the bridge. The sum S of the weights is approximately normally distributed with mean $3n$ (in thousands of pounds) and variance $(.09)n$. We want n such that $P(S \ge W) \ge .1$, i.e., $P(S - W \ge 0) \ge .1$. It is natural to assume that S

and D are independent, and so $S-W$ is normal with mean $3n-400$ and variance $(.09)n+40^2$. Then $P(S-W)\geq 0) = P(\frac{S-W-(3n-400)}{\sqrt{(.09)n+1600}} \geq \frac{-(3n-400)}{\sqrt{(.09)n+1600}})$ which is approximately $1-\Phi(\frac{-(3n-400)}{\sqrt{(.09)n+1600}})$. For this to be at least .1, we need $\Phi(\frac{-(3n-400)}{\sqrt{(.09)n+1600}}) < .9$. From the table, $\Phi(b) < .9$ for $b \leq 1.28$, and so the condition is that $\frac{-(3n-400)}{\sqrt{(.09)n+1600}} < 1.28$. This holds if $3n \geq 400$, since that is the condition for the right hand side to be ≤ 0. Now suppose that $3n < 400$. We want $400-3n < (1.28)\sqrt{(.09)n+1600}$. Square both sides to get $160000-2400n+9n^2 < (1.6384)((.09)n+1600) = (.147456)n+2621.44$ i.e., $9n^2-(2400.147465)n+157378.56 < 0$, with $n < 400/3$. The roots of the quadratic are 116.21 and 150.47 which means that it is negative between them. Hence, $n \geq 117$.

11. The change in price $Y = Y_{10}$ after 10 days should be approximately normal with mean 0 and $\sigma^2 = 10(1) = 10$. Then $P(Y \geq 5) = P(\frac{Y}{\sqrt{10}} \geq \frac{5}{\sqrt{10}}) = 1-\Phi(\frac{5}{\sqrt{10}}) = 1-\Phi(1.58) = 1-.9429 = .0571$. The probability should be $\approx .057$ — the book answer is wrong.

12. For the first part the sum is approximately normal with mean equal to the sum of the means which is $1000+(1/10)(1+\cdots+100) = 1000+(1/10)100(101/2) = 1505$, and variance equal to the sum of the variances (variance is mean squared here). The latter is the sum of the numbers $100+2i+i^2/100$ as i goes from 1 to 100 which is $10000+2(100)(101/2)+(1/100)(100)(101)(201)/6 = 10000+10100+3383.5 = 23483.5$. The variance is the square root, which is 153.24327 to five places, and so we want $1-\Phi(\frac{1200-1505}{153.24327}) = 1-\Phi(-1.99) = \Phi(1.99).9767$. In the second part each mean is the same, $(20+1/5)/2 = 10+i/10$, and the ith variance is $(20+i/5)^2/12 = (400+8i+i^2/25)/12 = (100/3)+(2/3)i+(1/300)i^2$. The sum of all these is $10000/3+(2/3)50(101)+(1/300)(100)(101)(201)/6 = 7827.88333$ which yields $\sigma \approx 88.475$. We want $1-\Phi(\frac{1200-1505}{88.475}) \approx 1-\Phi(-3.48) \approx \Phi(3.48) = .9997$.

13. Call the averages in the smaller and larger classes X and Y, resp. Both are approximately normal with mean 74, but with $\sigma = 14/5$ for X and $14/8$ for Y. $X-Y$ and $Y-X$ are both normal with mean 0 and $\sigma = \sqrt{((14/5)^2+(14/8)^2)} \approx 3.30189$. (a) $P(X > 80) = P((X-74)/(14/5)) > (80-74)/(14/5)) = P((X-74)/2.8) > 2.142857) \approx 1-\Phi(2.14) = 1-.9838 = .0162$. (b) is $\approx 1-\Phi((80-74)/(14/8)) \approx 1-\Phi(3.43) = 1-.9997 = .0003$. (c) and (d) have the same answer, $\approx 1-\Phi(2.2/3.30189) \approx 1-\Phi(.67) = 1-.7486 = .2514$.

14. If we have n components the sum of their lifetimes X has approximately normal distribution with mean $100n$ and standard deviation $30\sqrt{n}$. Then $P(X \geq 2000) = P(\frac{X-100n}{30\sqrt{n}} \geq \frac{2000-100n}{30\sqrt{n}})$ is approximately $1-\Phi(\frac{2000-100n}{30\sqrt{n}})$, and this is at least .95 if $\Phi(\frac{2000-100n}{30\sqrt{n}}) \leq .05$. From the table, $\Phi(1.65)$ is approximately .95, and so $\Phi(-1.65)$ is approximately .05. Thus, we want $\frac{2000-100n}{30\sqrt{n}} \leq -1.65$, which implies that $n > 20$. We assume $n > 20$ and multiply by -1, reversing the inequality. Thus, what we want is that $\frac{100n-2000}{30\sqrt{n}} \geq 1.65$ or that $100n-2000 \geq 30(1.65)\sqrt{n} = 49.5\sqrt{n}$ or $n-(.495)n^{1/2}-20 \geq 0$. The roots of the quadratic equation $u^2-(.495)u-20 = 0$ are -4.231, 4.726 which shows that the expression $n-(.495)n^{1/2}-20$ turns positive when $\sqrt{n}$ exceeds 4.726, and so we need $n \geq (4.726)^2 \approx 22.34$. Since n must be an integer we need $n \geq 23$.

15. The total claim is approximately normal with mean 2,400,000 and $\sigma = 800(100) = 80,000$. The answer is $1-\Phi(\frac{2700000-2400000}{80000}) = 1-\Phi(3.75)$, which is ≈ 0.