Math 425, Fall 1997
Prof. Hochster

Solutions to Selected Problems from Assignment #9

Routine details of integration are generally omitted.

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13. If we use minutes as the unit of time and consider the start of the hour as time 0, then the densities are 1/60 for the woman over [0,60] and 1/30 for the man over [15,45]. Let X be the woman's arrival time and Y the man's. Independence implies that the joint density is $\frac{1}{1800}$ over the rectangle $0 \leq x \leq 60$, $15 \leq y \leq 45$. We want $P(X-5 \leq Y \leq X) + P(X \leq Y \leq X+5) = P(X-5 \leq Y \leq X+5)$. This can be evaluated in several ways. E.g., it is $\int_{15}^{45} \int_{x-5}^{x+5} \frac{1}{1800}\, dy\, dx$. It can be calculated geometrically as $\frac{1}{1800}$ times the area of a parallelogram with base 10 and height 45-15 = 30. Or one can take a different approach altogether. No matter when the man arrives, we are asking for the probability that the woman will arrive in the ten minute interval centered at his arrival time, which is $10/60 = 1/6$. Since this is independent of when the man arrives, the answer is 1/6. (This is true even if his time of arrival is distributed in some other way over [15,45].) Note that we cannot condition on the woman's time of arrival: if she arrives before time 20, or after time 40, the length of the interval during which the man can arrive depends on the woman's arrival time.

Part (b) is similar: one can set up a double integral, or note instead that the line $x = y$ cuts the area of the rectangle in half, so that the answer must be 1/2.

15. (c) The probability is the area of the circle, $\pi(1)^2$, divided by the area of the region, which is 4, i.e., $\pi/4$.

20. In the first part the joint density can be written as the product of the individual densities: $(xe^{-x})(e^{-y})$. Independence follows.

In the second part, We may calculate $f_X(x) = \int_x^1 2\, dy = 2-2x$ while $f_Y(y) = \int_0^y 2\, dx = 2y$. Their product does not give the joint density, so that X and Y are not independent.

21. It is a joint density since it is nonnegative and $\int_0^1 \int_0^{1-x} 24xy\, dy\, dx = 1$. Then $E[X] = \int_0^1 \int_0^{1-x} (x)24xy\, dy\, dx = 2/5$ and $E[Y] = \int_0^1 \int_0^{1-x} (y)24xy\, dy\, dx = 2/5$. It is clear from symmetry that $E[Y] = E[X]$.

27. From independence the joint density is $(1)(e^{-y})$ for $0 < x < 1$ and $y > 0$ and 0 elsewhere. The distribution function for $X+Y$ is $P(X+Y \leq a)$ as a function of a. There are three cases. If $a \leq 0$ it is 0. If $0 < a \leq 1$ we need to integrate the density over the triangle formed by the axes and the line $x + y = a$, and we get $\int_0^a \int_0^{a-x} e^{-y}\, dy\, dx = a - 1 + e^{-a}$. If $a > 1$ the region formed by the axes, the line $x = 1$ and below the line $x + y = a$ is a trapezoid, and we get $\int_0^1 \int_0^{a-x} e^{-y}\, dy\, dx = 1 - e^{1-a} + e^{-a} = 1 - (e-1)e^{-a}$.

If $Z = X/Y$ the distribution function again vanishes for $a \leq 0$, and for $a > 0$ we want $P(X/Y \leq a) = P(Y \geq X/a) = \int_0^1 \int_{x/a}^{\infty} e^{-y}\, dy\, dx = a(1 - e^{-1/a})$.

33. Let X be Jill's score and Y be Jack's. Then $Y - X$ is approximately normal with mean $160 - 170 = -10$ and standard deviation $\sqrt{(20^2 + 15^2)} = 25$, and it follows that we have that $P(Y > X) = P(Y - X > 0) = P(\frac{Y-X-(-10)}{25} > \frac{0-(-10)}{25}) = P(N > 2/5)$ where N is standard normal. So this is $1 - \Phi(.4)$. (Since the scores are restricted to be integers it may be better to use .5 instead of 0, which gives a slightly different answer: $1 - \Phi(.42)$.) For the second part, note that $X + Y$ is approximately normal with mean 330 and standard deviation 25, so we want $P(X + Y > 350.5) = P(\frac{X+Y-330}{25} > \frac{350.5-330}{25})$ which is $1 - \Phi(.82)$.

43. One can determine c by integrating over the whole plane to see which value of c makes the integral 1 (and this gives $c = 1/8$), but it is not necessary to do this to work the problem. The conditional density $f_{Y|X}(y|x) = f(x,y)/f_X(x)$. The denominator is $\int_{-x}^{x} c(x^2 - y^2)e^{-x}\,dy = c(4/3)x^3e^{-x}$, so the conditional density is $\frac{x^2-y^2}{(4/3)x^3}$ (the c's cancel) or $\frac{3}{4}(x^{-1} - y^2x^{-3})$, provided that $-x \le y \le x$, and 0 otherwise. Then the conditional distribution $P(Y \le y \mid x)$ is 0 if $y < -x$, and is $\int_{-x}^{y} \frac{3}{4}(x^{-1} - y^2x^{-3})\,dy = \frac{1}{2} + \frac{3y}{4x} - \frac{y^3}{4x^3}$ **if** $-x \le y \le x$, while is 1 if $y > x$.

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5. The expected distance traveled will be the same for all four squares or quadrants formed by drawing perpendicular lines through the center parallel to the sides: consider one quadrant, which we may think of as the square $0 \le x \le 1.5, 0 \le y \le 1.5$. Then $E[X + Y] = E[X] + E[Y] = (1.5/2) + (1.5/2) = 1.5$, since each of X, Y is uniformly distributed over $[0, 1.5]$ for this quadrant.

6. Let X_i be the expected value of the die on the ith roll, which is $1/6+2/6+3/6+4/6+5/6+6/6 = 21/6 = 3.5$. Then $E[X_1 + \cdots + X_{10}] = E[X_1] + \cdots + E[X_{10}] = 10(3.5) = 35$.

7. (a) Let $X_i = 1$ if the ith object is chosen by both and 0 otherwise. $E[X_i]$ is the probability that the ith object will be chosen by both, $(.3)(.3) = .09$. $E[X_1 + \cdots X_{10}] = 10(.09) = .9$. (b) is similar: then answer is $10(.7)(.7) = 4.9$, since the probability that a given object will be chosen by neither is $(.7)(.7)$. Likewise, (c) is $10\big((.3)(.7)+(.7)(.3)\big) = 4.2$.

12. Let X_i be 1 if there is at least one woman next to the ith man and 0 otherwise. The sum of the n numbers $E[X_i]$ is n times the probability that there is at least one woman next to the ith man. We condition on whether the man is at an end spot or not. The answer is $n\Big((\frac{2}{n+m})(\frac{m}{n+m-1}) + (\frac{n+m-2}{n+m})(1 - \frac{\binom{n-1}{2}}{\binom{n+m-1}{2}})\Big)$. This can be rewritten as $n\frac{m^2+2mn-m-2n+2}{(m+n)(m+n-1)}$.

18. Let X_i be 1 if the ith card matches the number i and 0 otherwise. Then $E[X_i] = 4/52$, and the sum of the 52 numbers $E[X_i]$ is $52(4/52) = 4$.