Math 425, Fall 1997
Prof. Hochster

Solutions and Comments for Selected Problems from Assignments #4 & #5

These problems are from pages 173–179 of the text. In some cases I have only made a comment, in others I have given most of the solution, while most problems are worked out in complete detail.

#15. By the rules given, the team with the worst record gets the number 4 pick if it does not get one of the first three, so the random variable takes on only the values 1, 2, 3, and 4. Let p_i be the probability that the worst team gets the i th pick. Then $p_4 = 1 - p_1 - p_2 - p_3$ and the expeced number of the pick is $1p_1 + 2p_2 + 3p_3 + 4p_4$. The problem is to calculate p_1, p_2 and p_3. First, $p_1 = 11/66 = 1/6 = .16667$ to five decimal places, since the worst team has 11 of the 66 balls in the urn. To get p_2, condition on the number of balls i in the urn of the team that does get the first pick, if the worst does not. This leads to the conclusion that p_2 is the sum of the terms $(i/66)(11/(66 - i) = i/6(66 - i)$ as i varies from 1 to 10. (The situation after the first pick is the same as if all the balls for the team that gets the first pick are removed immediately from the urn, since they are discarded if picked after that). Using a programmable calculator or Maple, one gets that $p_2 = .15563$. Similarly, by conditioning on the numbers i, j of balls in the urn of the first two teams picked, if neither is the worst team, one sees that p_3 is the sum of all the terms $(i/66)(j/(66-i))(11/(66-i-j)) = ij/(6(66-i)(66-i-j))$ as i, j run through all distinct values between 1 and 10 (ninety terms). I used the following Maple script to calculate this:

```
> x := 0.0;
> for i from 1 to 10 do
> for j from 1 to 10 do
> if j <> i then x := x + i*j/(6*(66-i)*(66-i-j));
> fi;
> od;
> od;
> x;
```

This gives $p_3 = .1434756999$ or .14348 to five decimal places and so we obtain that $p_4 = 1 - .16667 - .15563 - .14348 = .53422$. Therefore the expected number of the pick is $1(.16667) + 2(.15563) + 3(.14348) + 4(.53422) = 3.04525$.

#17. As $b \to 1$ from the left, $F(b) \to 1/4$, which is $P(X < 1)$, while $P(X \leq 1) = F(1) = 1/2$. Thus, $P(X = 1) = 1/2 - 1/4 = 1/4$. Similarly, $P(X < 2)$ is the limit of $F(b)$ as $b \to 2$ from the left, which is $3/4$, and $P(X \leq 2) = F(2) = 11/12$, so that $P(X = 2) = 11/12 - 3/4 = 2/12 = 1/6$, and $P(X < 3) = 11/12$ while $P(X \leq 3) = 1$, so that $P(X = 3) = 1/12$. Also, $P(X < 3/2) = P(X \leq 3/2) = F(3/2) = 5/8$, since F is continuous at $3/2$, and $P(X \leq 1/2) = F(1/2) = 1/8$. Then we have $P(1/2 < X < 3/2) = P(X < 3/2) - P(X \leq 1/2) = 5/8 - 1/8 = 1/2$.

#21. If a student is selected the expected value of the number of students in that student's bus is $40(40/148)+33(33/148)+25(25/148)+50(50/148) = (1600+1089+625+2500)/148 =$

$5814/148 = 39.28378$ to five decimal places. If the driver is selected at random the expected value is $40(1/4) + 33(1/4) + 25(1/4) + 50(1/4) = 148/4 = 37$.

#23. Suppose you line up the components to be tested in a certain order, 1, 2, 3, 4, 5. There are ten equally likely pairs of spots that the defectives might be in:

12
13, 23
14, 24, 34
15, 25, 35, 45

For the first row, only two tests are needed. For the second, three are needed. For the third, four are needed. But the problem is ambiguous — for the cases in the fourth row logic indicates that if only one defective has been found after four tests, then the fifth is the other. However, one might want to do the fifth test anyway, to make sure.

In the first interpretation the answer is $2(1/10) + 3(2/10) + 4(7/10) = 3.6$ while in the second the answer is $2(1/10) + 3(2/10) + 4(3/10) + 5(4/10) = 4$.

#25. The problem is ambiguous as stated, because it is not clear whether the payoff is "net," or not, i.e., whether, for example, the 60 in the first row (of the second table) includes the loss of the coin you put in to play. The implication of the final row of the payoff table would seem to be that it does, but if the loss of the coin is not taken into account in the other rows, then the "net" payoffs are 59, 19, 17, etc. and the expectation becomes smaller.

#31. Here, treat p^* as fixed and p as variable. Given that the meterologist believes that it will rain with probability p^*, the meteorologist's expectation with the answer "p" is $p^*(1-(1-p)^2)+(1-p^*)(1-p^2) = p^*(2p-p^2)+1-p^*-(1-p^*)p^2 = (1-p^*)+(2p^*)p-p^2$. We want to maximize this on the interval $0 \le p \le 1$, treating p^* as constant. The derivative is 0 when $2p^* - 2p = 0$, i.e., at $p = p^*$. This will be in $[0, 1]$, and since the leading coefficient is negative this is a parabola opening downward, so that the absolute maximum occurs where the derivative is 0, and is strictly greater than the value anywhere else. Therefore, the meteorologist should predict that it will rain with probability p^*.

#33. Let E_b be the expected profit from the bth paper sold, in cents. Then $E_b = -10 + 15P(X \ge b)$ since a dime is lost from buying the paper, while 15 cents is gained only if the demand X is at least b. The total expectation is the sum of the E_i, and since $P(X \ge i)$ decreases when i increases, the E_i are positive for a while and then eventually become and remain negative. E is maximum when the number of papers bought is chosen so that all the E_i are positive or, at least, nonnegative. But we have that $E_b \ge 0$ iff $P(X \ge b) \ge 10/15$, and this is equivalent to $P(X \le b-1) \le 5/15$. Now $p_i = P(X = i) = \binom{10}{i}(1/3)^i(2/3)^{10-i} = \binom{10}{i}2^{10-i}/3^{10}$. Then $p_0 = (2/3)^{10} = .01734$ (to five decimal places), $p_1 = 10(2^9/3^{10}) = .08671$, $p_2 = 45(2^8/3^{10}) = .19509$, $p_3 = 120(2^7/3^{10}) = .26012$. Thus $p_0 + p_1 + p_2 = .29914 < 5/15$, while adding p_3 produces a total larger than $5/15$. Thus, $b - 1 = 2$, i.e., $b = 3$ produces the largest expected profit. Notice that the ratio $10/15$, i.e. cost/price, has a big effect on the number of papers one should buy to maximize expected profit — if it is very small, one might want to buy ten papers. The fact that $E(X)$, which, by pure coincidence, is $10/3$ and is close to 3, is not particularly relevant here. It is also a coincidence that $10/15 = 2/3 = 1 - p$.

#35. The probability that the balls will match is $(\binom{5}{2}+\binom{5}{2})/\binom{10}{2} = (10+10)/45 = 4/9$. The expected value is therefore $1.10(4/9)+(-1)5/9 = -.6/9 = -1/15 = -.06667$ cents to 5 decimal places. The variance is $E[X]^2 - E[X]^2 = (1.21)(4/9) + 1(5/9) - (1/15)^2 = 484/900 + 5/9 - 1/225 = (484+500-4)/900 = 980/900 = 49/45 = 1.08889$ to five places.

#37. (based on #21.) If the student is selected, then $E[X^2] = 40^2(40/148)+33^2(33/148)+25^2(25/148)+50^2(50/148) = (64000+35937+15625+125000)/148 = 240562/148 = 1625.41892$ to five decimal places, and the variance is then $625.41892 - (39.28378)^2 = 82.20355$ to five decimal places.

If the driver is selected then $E[Y^2] = 40^2(1/4)+33^2(1/4)+25^2(1/4)+50^2/(1/4) = (1600+1089+625+2500)/4 = 5814/4 = 1453.5$ (exactly) and the variance is $1453.5 - 37^2 = 1453.5 - 1369 = 84.5$ (exactly).

#38. $\mathrm{Var}(X) = E[X^2] - E[X]^2$ and so $5 = E[X^2] - 1$, so that $E[X^2] = 6$. Then $E[(X+2)^2] = E[X^2+4X+4] = E[X^2]+4E[X]+4 = 6+4(1)+4 = 14$, while $\mathrm{Var}(4+3X)$ may be calculated either from the formula on p. 144 as $3^2\mathrm{Var}(X) = 9(5) = 45$ or as $E[(3X+4)^2] - E[3X+4]^2 = E[9X^2+24X+16] - (3E[X]+4)^2 = 9E[X^2]+24E[X]+16-(3\cdot 1+4)^2 = 9(6)+24+16-7^2 = 54+24+16-49 = 45$.

#39. $\binom{4}{2}(1/2)^2(1/2)^2 = 6/16 = 3/8 = .375$.

#41. Note the words "at least this well." X is binomial with $n = 10$ and $p = 1/2$. This is $p(7)+p(8)+p(9)+p(10)$, **not** just $p(7)$. So this is $\binom{10}{7}/2^{10}+\binom{10}{8}/2^{10}+\binom{10}{9}/2^{10}+\binom{10}{10}/2^{10} = (\binom{10}{3}+\binom{10}{2}+\binom{10}{1}+\binom{10}{0})/2^{10} = (120+45+10+1)/1024 = 176/1024 = .1718175$ (exactly).

#51. It is reasonable to expect the number of errors per page to be well approximated by a Poisson random variable X with $\lambda = .2$. Then $P(X=0) = e^{-.2}$, $P(X=1) = e^{-.2}(.2)/1! = .2e^{-.2}$, and $P(X \geq 2) = 1 - P(X=0) - P(X=1) = 1-(1.2)e^{-.2}$.

#52. It is reasonable to expect the number of crashes per month to be well approximated by a Poisson random variable X with $\lambda = 3.5$. Then $P(X=0) = e^{-3.5}$, $P(X=1) = e^{-3.5}(3.5)/1! = (3.5)e^{-3.5}$, and $P(X \leq 1)$ is the sum of these two, $(4.5)e^{-3.5}$, while, finally, $P(X \geq 2) = 1 - P(X=0) - P(X=1) = 1-(4.5)e^{-3.5}$.

#54. It is reasonable to expect the number of cars abandoned per week to be well approximated by a Poisson random variable X with $\lambda = 2.2$. Then $P(X=0) = e^{-2.2}$ (this is part (a)), $P(X=1) = e^{-2.2}(2.2)/1! = (2.2)e^{-2.2}$, and $P(X \leq 1)$ is the sum of these two, $(3.2)e^{-2.2}$, while $P(X \geq 2) = 1 - P(X=0) - P(X=1) = 1-(3.2)e^{-2.2}$.

#58. Note that in part (b) the two values are not close: this is because p is large.

#60. The probability that a person gets two colds is $p+q$ where $p = (.75)e^{-3}3^2/2!$ is the probability that the person is in the group that benefits from the drug and gets two colds and $q = (.25)e^{-5}5^2/2!$ is the probability that the person is in the group that does not benefit from the drug and gets two colds. The probability that the drug is beneficial is $p/(p+q)$.

#63. It is reasonable to approximate the number of people entering by a Poisson random variable. Since the average rate is 1 in two minutes, for a five minute interval $\lambda = 5/2 = 2.5$. Thus, $p_0 = e^{-2.5}$, $p_1 = e^{-2.5}(2.5)/1!$, $p_2 = e^{-2.5}(2.5)^2/2!$, $p_3 = e^{-2.5}(2.5)^3/3!$ and $P(X \geq 4) = 1 - p_0 - p_1 - p_2 - p_3$.