We construct a one-dimensional ring R whose local rings are all fields and discrete Noetherian valuation domains such that Spec$\,(R)$ is connected but R is not a domain.

Let H be an infinite totally ordered set with the property that between any two distinct elements of H there is another element. Let $\{x_h : h \in H\}$ be a family of indeterminates indexed by H: order them so that $x_h < x_k$ precisely when $h < k$. We form a commutative multiplicative semigroup S whose elements consist of 1 and the positive powers of the individual x_h. The multiplication is given by the rule that $x_h^m x_k^n$ is
(1) x_h^m if $h < k$ (2) x_h^{m+n} if $h = k$ (3) x_k^n if $k < h$ (forced from (1) by commutativity).

The operation is easily checked to be commutative and associative: no matter how one inserts parentheses, $x_g^m x_h^n x_k^r$ will be the least of the three variables occurring, with the exponent that is the sum of the exponents with which it occurs in the product. It is easy to verify this by considering the three cases determined by the number of occurrences of the smallest variable among the three terms.

Now let K be a field, and let R be the semigroup ring of S over K. We shall show that R has connected spectrum and is locally a domain but not a domain. We first show that R contains no idempotents except 0, 1 (thus, Spec$\,(R)$ is connected) and no nilpotent except 0. Suppose one had such an element r. It cannot be a constant. Consider the highest degree terms that occur, and from among them pick the one with the largest variable: suppose that this term is cx_h^d, where $c \in K - \{0\}$. Then the expansion of r^2 involves a term $c^2 x_h^{2d}$ and only one such term occurs: it cannot be canceled. This term shows that $r^2 \neq 0$, and also that $r^2 \neq r$, since $\deg(r^2) > d$.

Evidently R is not a domain, since whenever $h < k$, $x_h = x_h x_k$ and $x_h(1 - x_k) = 0$. (These relations generate the ideal of relations on the x_h, although we do not need this.) It remains to show that R is locally a domain, and so we study the prime ideals of R. Call $J \subseteq H$ an *upper* (respectively, *lower*) interval if whenever $h \in J$ and $k > h$ (respectively, $k < h$) then $k \in J$ as well.

Given a prime ideal P, let J_P denote the subset of H consisting of those h such that $x_h \notin P$. Note that if $x_h \notin P$ then for all $k > h$, $x_h(1 - x_k) = 0$ implies that $1 - x_k \in P$, and so $k \in J_P$ as well. Hence, J_P is an upper interval in H. Evidently, the set I_P of h such that $x_h \in P$ is a lower interval. Note that I_P, J_P give a partition of H such that every element of I_P is less than every element of J_P. We now consider what happens when we localize at P. If k is an element of J_P but not a least element, then we can choose $x_h \notin P$ with $h < k$, and the equation $x_h(1 - x_k) = 0$ forces x_k to be identified with 1 in the localization for all k except possibly the least element in J_P. If g is any element of I_P other than the greatest, we can choose $h \in I_P$ with $g < h$, and then the equation $x_g(1 - x_h) = 0$ forces x_g to become 0 in the localization. Our assumption on the totally ordered set implies that either I_P has no greatest element, or J_P has no least element.

If there is no greatest element in I_P and no least element in J_P the localization is K. If x_h is greatest in I_P or least in J_P, all other x_k are identified with 0 or 1 after localizing at P. The local ring one gets is a localization of the polynomial ring $K[x_h]$. This shows that R is locally a domain and has Krull dimension 1. $\square$