Math 614, Fall 2008 **Problem Set #1**
Due: Friday, September 26

In **#1.** and **#2.** K is a field and $R = K[x]$ is a polynomial ring in one variable over K.

1. (a). Show that if $h,\, k > 0$ are integers with greatest common divisor d, then the subring $K[x^h,\, x^k]$ of R generated over K by x^h and x^k contains x^{dn} for all integers $n \gg 0$.

(b) Show, more generally, that if $h_1, \ldots, h_s$ are positive integers with greatest common divisor d, the subring $K[x^{h_1}, \ldots x^{h_s}]$ of $K[x]$ generated over K by $x^{h_1}, \ldots x^{h_s}$ contains x^{nd} for all integers $n \gg 0$.

2. (a) Show that for every integer $n > 0$, x^{2n-1}, $x^{2n} + x$, and x^{2n+1} generate $K[x]$ over K.

(b) Consider the subring $T = K[x^2,\, x^3 + x] \subseteq K[x]$. Let $A = K[x^2]$. Show that every element of T can be written uniquely in the form $f + (x^3 + x)g$, where $f,\, g \in A$. Conclude that every element of T of odd degree has degree at least 3. Hence, $x \notin T$ and $T \neq K[x]$.

(c) Show that x *is* in the field of fractions of T.

3. Let R be a commutative ring and let $u \neq 0$ be an element of R that is in every nonzero ideal of R. Let I be the annihilator of u, i.e., $I = \{r \in R : ru = 0\}$. Prove that I is maximal, and that if $f \in R - I$, then f is a nonzerodivisor on R, i.e. if $fz = 0$ then $z = 0$.

4. Let R be any commutative ring with identity.

(a) Let I be an ideal of R. Let $h : R \to R/I$ be the quotient homomorphism. Hence, $\mathrm{Spec}\,(h) : \mathrm{Spec}\,(R/I) \to \mathrm{Spec}\,(R)$ is continuous. Show that the image of this map is $V(I) = \{P \in \mathrm{Spec}\,(R) : P \supseteq I\}$, and that if the range is restricted to $V(I)$, the map yields a homeomorphism $\mathrm{Spec}\,(R/I) \cong V(I)$.

(b) Let $\mathcal{P}$ denote a family of prime ideals in the commutative ring R. Let I be the intersection of the ideals in this family. Prove that closure of $\mathcal{P}$ in $\mathrm{Spec}\,(R)$ is $V(I)$.

5. Let X be a compact (i.e., quasicompact Hausdorff) space. You may assume that such a space is *normal* (i.e., T_4), so that disjoint closed sets have disjoint open neighborhoods. Hence, a continuous $\mathbb{R}$-valued function on a closed set $Z \subseteq X$ extends continuously to all of X (the Tietze extension theorem). Let $\mathcal{C}(X) = \{f : X \to \mathbb{R} : f \text{ is continuous}\}$.

(a) Prove that there is a bijection between the maximal ideals of $\mathcal{C}(X)$ and the points of X, where the maximal ideal m_x corresponding to $x \in X$ is $\{f \in \mathcal{C}(X) : f(x) = 0\}$.

(b) Prove that if we give $\mathrm{MaxSpec}\big(\mathcal{C}(X)\big)$ the inherited Zariski topology, the bijection mapping x to m_x is a homeomorphism.

6. Let $\mathcal{C}$ be the category of R-modules for some ring R, and let W be a fixed R-module. Let $D = h^W$ denote the contravariant functor from $\mathcal{C}$ to $\mathcal{C}$ whose value on M is $\mathrm{Hom}_R(M,\, W)$. Then $D \circ D$ is a covariant functor from R-modules to R-modules. Let I denote the identity functor on R-modules. Show that there is a natural transformaton $T : I \to D \circ D$ whose value T_M on the R-module M is the map $M \to D\big(D(M)\big)$ such that, for all $u \in M$, $T_M(u)$ is the map $D(M) \to W$ that sends $L \in D(M)$ to $L(u)$. (When $R = K$ is a field and $W = K$, this natural transformation was discussed in class.)