Math 614, Fall 2008 **Problem Set #2**
Due: Wednesday, October 15

1. An object X in a catgory $\mathcal{C}$ is called an *initial object* in $\mathcal{C}$ if for every object Y of $\mathcal{C}$ there is a unique morphism from $X \to Y$. X is called a *terminal* object in $\mathcal{C}$ if for every object Y in $\mathcal{C}$ there a unique morphism from Y to X (equivalently, if X is an initial object in $\mathcal{C}^{\mathrm{op}}$).

(a) Determine whether there exists an initial object (respectively, a final object) in the category of sets and functions.

(b) Determine whether there exists an initial object (respectively, a final object) in the category of commutative rings with identity and ring homomorphisms that preserve the identity.

(c) Let R be a commutative ring. Determine whether there is an initial object (respectively, a final object) in the category of R-modules and R-linear maps.

2. Let W be a multiplicative system in R, let $S = W^{-1}R$, and let T be another ring. Let $f : R \to S$ be the usual map ($r \mapsto r/1$). Let g and h be ring homomorphisms from S to T. Prove carefully the assertion made in class that if $g \circ f = h \circ f$ then $g = h$. (That is, f is an epimorphism.)

3. Let W and V be multiplicative systems in R. Let $S = W^{-1}R$, and let $\widetilde{V}$ be the image of V in S. Let U be the multiplicative system $VW = \{vw : v \in V,\, w \in W\}$. Prove carefully that $\widetilde{V}^{-1}S \cong U^{-1}R$ as R-algebras, making use of the universal mapping property of localization to construct the homomorphisms needed.

4. Let k be a positive integer such that $p = 4k + 1$ is prime. Show that the ring $\mathbb{Z}[\sqrt{p}]$ is not normal, and show that its normalization consists of all elements of the form $\mathbb{Z} + \mathbb{Z}s$ for some element s in its fraction field. Give s explicitly.

5. Let R be any commutative ring.

(a) Let Q and Q' be prime ideals of R. Show that there is a prime ideal P that is contained in $Q \cap Q'$ if and only if there do not exist elements $a \in R - Q$ and $b \in R - Q'$ such that $ab = 0$. Conclude that if there is no prime ideal contained in $Q \cap Q'$, then Q and Q' have disjoint open neighborhoods.

(b) Let R have Krull dimension 0. Show that $X = \mathrm{Spec}\,(R)$ is a compact Hausdorff space. Conclude that every quasicompact open set is closed, and that if x, y are distinct points of X, then there is a clopen set (closed and open set) containing x and not y.

6. Let R be a ring and let $S = R[x_1, \ldots, x_n]$ be the polynomial ring in n variables over R. For $1 \le i \le n$ let $F_i \in S$ be a polynomial of degree n_i whose only term of degree n_i is $x_i^{n_i}$. Prove that $S/(F_1, \ldots, F_n)S$ is module-finite over R.