Math 614, Fall 2008 **Problem Set #3**
Due: Monday, November 10

1. Let $f, g \in R$ be such that $f + g = 1$. Let $u \in R_f$ and $v \in R_g$ have the same image in R_{fg}. Show there is a unique $r \in R$ whose image in R_f is u and whose image in R_g is v.[1]

2. Let $S = R[x]$ be a polynomial ring in one variable over R, and let $P \subseteq R$ be prime. Let $\kappa = \mathrm{frac}(R/P)$, the fraction field of R/P. Note that we have $R[x] \twoheadrightarrow (R/P)[x] \hookrightarrow \kappa[x]$. Show that the prime ideals Q of S lying over P have one of two types as follows: either $Q = PR[x]$, or there is a polynomial $f(x)$ with leading coefficient in $R - P$ such that the image $\overline{f}$ of f in $\kappa[x]$ is irreducible and Q is the contraction of $(\overline{f}) \subseteq \kappa[x]$ to $R[x]$.

3. R is called a *Hilbert ring* if every prime ideal of R is an intersection of maximal ideals.

(a) Show that a homomorphic image of a Hilbert ring is a Hilbert ring.

(b) Show that if R is a Hilbert ring and $f \in R$, the contraction of every maximal ideal of R_f to R is maximal. Also show that R_f is a Hilbert ring.

4. Let R be a Hilbert ring and S a finitely generated R-algebra.

(a) Show that S is a Hilbert ring. (It suffices to consider polynomial rings and, hence, $R[x]$. Let Q be a prime ideal of $R[x]$. The result of problem **2.** may be helpful in showing that Q is an intersection of maximal ideals of $R[x]$.)

(b) Show that if R is a domain, $R \subseteq S$, and S is a field, then R is a field.[2] [One can reduce to the case where S is generated over R by one element.]

(c) Show that every maximal ideal of S lies over a maximal ideal of R.

5. Let K be an algebraically closed field. This hypothesis continues in problem **6.** below.

(a) Let $f : K^2 \to K^2$ be the morphism such that $f(x, y) = (x, 1 + xy)$ for all $x, y \in K$. Find the image of f, and show that it is neither open nor closed in K^2.

(b) Let $g : K^2 \to K^2$ be such that $g(x, y) = \big(x + y(1 + xy), 1 + xy\big)$ for all $x, y \in K$. Find the image U of g, and show that it is open in K^2. Describe the sets $A_i \subseteq U$ of points P such that $g^{-1}(P)$ has i elements for $i = 1, 2$.

6. Let $P_1, \ldots, P_n$ be any n distinct points of K^2.

(a) Let $Q_1, \ldots, Q_n$ be any set of n distinct points of K^2. Prove that there is an isomorphism $f : K^2 \to K^2$ such that $f(P_i) = Q_i$, $1 \le i \le n$.

(b) Show that there is a morphism $g : K^2 \to K^2$ whose image is precisely $K^2 - \{P_1, \ldots, P_n\}$.

EXTRA CREDIT Is there a ring R with an element $u \neq 0$ such that u is in every ideal and $m = \mathrm{Ann}_R u = \{r \in R : ru = 0\}$ does not consists entirely of nilpotent elements? Is there such a ring R such that R contains a nonzerodivisor that is not a unit?

---

[1] This completes the proof of the argument, the rest of which was given in class, that a map of closed algebraic sets $X \to Y$ is regular if that is true locally on X.

[2] Part (a) shows that a finitely generated algebra over a field K is a Hilbert ring, since K is, and part (b) recovers Zariski's lemma and, hence, Hilbert's Nullstellensatz.