Math 614, Fall 2008 **Problem Set #4**

Due: Monday, December 1

1. Let $\mathbb{Z}_+$ be the set of positive integers. Let T be the polynomial ring $K[x,\,y]$ in two variables over a field K and let $S = K[x,\,y][x/y^n : n \in \mathbb{Z}_+]$, which is a subring of the fraction field of T. The elements x, y, and $\{x/y^n : n \in \mathbb{Z}_+\}$ generate a maximal ideal $\mathcal{M}$ of S. Let $J = x\mathcal{M}$, and let $R = S/J$. Let u be the image of x in R. Show that u is a nonzero but that u is in every nonzero ideal of R. Show that the annihilator of u in R is m/J. Show also that the ring $K[y]$ is a homomorphic image of R, so that R is not quasilocal. (Clearly, $y \in m$ is not nilpotent.)

2. Let R be a local ring, and let f be an element of R that is not a zerodivisor. Suppose that R/fR is an integral domain. Prove that R is an integral domain.

3. Let R be a ring and I, J ideals of R.

(a) Show that the kernel of the map $R/J \otimes_R I \to R/J$ obtain by applying $R/J \otimes_R \_$ to the inclusion $I \subseteq R$ is $(I \cap J)/IJ$.

(b) Show that if R/J is flat, then $I \cap J = IJ$ for every ideal I of R.

(c) Show conversely that if $I \cap J = IJ$ for every ideal I of R, then R/J is flat as an R-module.

4. Let K be a field, and let $Y,\, X_1,\, X_2,\, X_3,\, \ldots X_n,\, \ldots$ be countably indeterminates over K. Let $T = K[Y,\, X_1,\, X_2,\, X_3,\, \ldots X_n,\, \ldots]$ and let $J = (X_n y^n : n \geq 1)T \subseteq T$. Let $R = T/J$. Let Q denote the prime ideal of R generated by all the x_n, $n \geq 1$, so that $R/Q \cong K[y]$. Let y be the image of Y in R. Show that $\mathrm{Hom}_R(R/Q,\, R) = 0$, while $\mathrm{Hom}_{R_y}\big((R/Q)_y,\, R_y\big) \neq 0$. Hence, localization at $W = \{y^n : n \in \mathbb{N}\}$ does not commute with Hom_R in this instance. [Of course, R/Q is not finitely presented, although it is finitely generated.]

5. Let R be a ring and M an R-module.

(a) Prove that M is flat if and only if M_P is R_P-flat for every prime ideal P of R.

(b) Suppose that R is reduced and 0-dimensional. Prove that every R-module is flat.

(c) Let R be a ring such that every R-module is flat.

(1) Prove that for every element $r \in R$ there exists an element $u \in R$ such that $r = ur^2$. Conclude that $V(r)$ is open as well as closed.

(2) Prove that R is reduced.

(3) Prove that the Krull dimension of R is 0.

6. Let R be a commutative ring. Suppose that every local ring R_P of R is an integral domain and that $\mathrm{Spec}\,(R)$ is connected. Show that if R has only fnitely many minimal primes, then R is an integral domain.

EXTRA CREDIT Suppose that the ring R satisfies the hypotheses in the second sentence of **6.**, but that no condition is imposed on the minimal primes of R. Must R be an integral domain?