Math 614, Fall 2008 **Problem Set #5**
Due: Monday, December 15

1. Let M and N be modules of finite length over the Noetherian ring R. Show that $M \otimes_R N$ and $\mathrm{Hom}_R(M,\, N)$ have finite length. Also show that a descending sequence of cosets $m_n + Q_n$ of submodules $Q_n \subseteq M$ is eventually constant.

2. Let $M \neq 0$ be finitely generated over a Noetherian ring R with $\mathrm{Ass}\,(M) = \{P_1,\, \ldots\,, P_n\}$.
(a) Let P be maximal in $\mathrm{Ass}\,(M)$. Let $N = \mathrm{Ann}_M P$. Prove that $N \neq 0$ is a torsion-free module over the domain R/P, and $\mathrm{Ass}\,(M/N) \subseteq \{P_1,\, \ldots\,, P_n\}$.
(b) Prove that M has a finite filtration in which each factor N is a nonzero torsion-free module that is killed one of the P_i. Moreover, show that every P_i occurs.

3. Let R, M be as in **2.**. Let S be a Noetherian R-flat algebra. Show that $\mathrm{Ass}_S(S \otimes_R M) = \bigcup_{i=1}^n \mathrm{Ass}_S(S/P_iS)$. Conclude that If $S = R[x_1,\, \ldots\,, x_k]$ is polynomial over R, then $\mathrm{Ass}_S(S \otimes_R M) = \{PS : P \in \mathrm{Ass}_R(M)\}$.

4. (a) Let m be a maximal ideal of a ring R. Let $\mathcal{M} = mR_m$. Show that $\widehat{R}^m \cong \widehat{R_m}^{\mathcal{M}}$.
(b) Let $S = K[x,\, y]$ be the polynomial ring in two variables over a field K. Let $T = K[[x,\, y]]$ be its $(x,\, y)$-adic completion. Let $f = y^2 - x^2 - x^3$. Show that $R = K[x, y]/fS$ is a domain but that its completion T/fT with respect to the maximal ideal $(x,\, y)R$ is not.
(c) Let $f = \sum_{n=1}^{\infty} x^{n!} \in K[[x]]$, the formal power series ring in one variable over the field K. Show that that f is transcendental over $K[x]$.

5. Let $R = K[x, y]$ be a polynomial ring in two variables over K and let $S = K[[x, y]]$ be its $(x,\, y)$-adic completion. Let $g \in xK[[x]] \subseteq S$ be transcendental over $K[x]$, and suppose $g = \sum_{t=1}^{\infty} c_t x^t$ with the $c_t \in K$. Let $g_n = \sum_{t=1}^{n-1} c_t x^t$. Let $I_n = (y - g_n, x^n)R$, and note that $I_nS = (g,\, x^nS)$. Prove that $\bigcap_n I_nS \neq (\bigcap_n I_n)S$. (Show that the left hand side is $(y - g)S$, and that $(y - g)S \cap R = (0)$, which implies that the right hand side is (0).)

6. Let $\{A_n\}_{n\in\mathbb{N}}$ be an inverse limit system over the nonnegative integers $\mathbb{N}$, where the $A_n \neq \emptyset$ are sets, groups, rings, or modules over a fixed ring R.
(a) Show that if the maps $A_{n+1} \to A_n$ are onto then $L = \varprojlim_n A_n$ is not empty.
(b) If, for each fixed n, $\mathrm{Im}(A_N \to A_n)$ is constant for all $N \gg 0$, show $L \neq \emptyset$.
(c) Let $\{A_n\}_{n\in\mathbb{N}}$, $\{B_n\}_{n\in\mathbb{N}}$, and $\{C_n\}_{n\in\mathbb{N}}$ be inverse limit systems of modules such that the for all n one has an exact sequence $A_n \to B_n \to C_n$. Suppose also that for all n the diagram

$$\begin{array}{ccccc} A_{n+1} & \longrightarrow & B_{n+1} & \longrightarrow & C_{n+1} \\ \downarrow & & \downarrow & & \downarrow \\ A_n & \longrightarrow & B_n & \longrightarrow & C_n \end{array}$$

commutes. *Suppose that all the modules have finite length.* Prove that $\varprojlim_n A_n \to \varprojlim_n B_n \to \varprojlim_n C_n$ is exact.

EXTRA CREDIT Let $(*)\quad 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be an exact sequence of modules of finite length over the Noetherian ring R. Suppose that $B \cong A \oplus C$. Prove $(*)$ is split.