Math 614, Fall 2008 **Supplementary Problem Set #1**

1. (a) Let K be a field and x an indeterminate over K. Show that every subring R of $K[x]$ that contains K is finitely generated over K. (Suggestion: if $f \in R - K$ has degree $d > 0$, show that $K[x]$ is spanned as a module over $K[f]$ by $1, x, \ldots, x^{d-1}$. A submodule of a finitely generated module over a PID is finitely generated, even as a module.)

(b) Show that $x^3, x^4 + x, x^5$ generate $\mathbb{C}[x]$.

2. Show that the Zariski topology on $\mathrm{Spec}\,(R)$ is quasi-compact, i.e., that if a family of closed sets has the property that any finite subfamily has non-empty intersection, then the entire family has non-empty intersection.

3. Let R be the ring of all functions from the non-empty set X to the field K. Prove for $f, g, h \in R$, that $g \in fR$ if and only if $g^{-1}(0) \supseteq f^{-1}(0)$ and that $h \in (f, g)R$ if and only if $h^{-1}(0) \supseteq f^{-1}(0) \cap g^{-1}(0)$. Show that every finitely generated ideal is principal. Prove the bijection between ideals of R and filters on X stated in the lecture of September 5. Show that this ring is Noetherian if and only if X is finite.

4. Suppose that X and Y are objects of a category $\mathcal{C}$ such that the functors h_X and h_Y are isomorphic functors. Is it necessarily true that $X \cong Y$ in $\mathcal{C}$?

5. Let R be a commutative ring and let S be a multiplicative system in R. Let $T = S^{-1}R$. Suppose that M and N are T-modules: they are also R-modules by restriction of scalars. Show that $\mathrm{Hom}_T(M, N) = \mathrm{Hom}_R(M, N)$, i.e., that an R-linear map from M to N is automatically T-linear.

6. An element e in a ring R is called *idempotent* if $e^2 = e$. Note that e is idempotent if and only if $1 - e$ is idempotent, since the condition may be written $e(1 - e) = 0$. If $R \cong S \times T$, where S and T are nonzero rings, then $e = (1_S, 0)$ and $1 - e = (0, 1_T)$ are non-trivial idempotents in R, where an idempotent is *non-trivial* if it is different from 0 and 1.

(a) Show conversely that if R has a non-trivial idempotent e that $R \cong S \times T$ where $S = Re$ has identity e and $T = R(1-e)$ has identity $f = 1-e$, and that both of these factor rings are non-trivial. Thus, non-trivial product decompositions of R correspond to non-trivial idempotents in R.

(b) Let N be the ideal of all nilpotent elements of R. Suppose that R/N has an idempotent element e that is different from 0 and 1. Show that there is a non-trivial idempotent e^* of R such that $e^* + N = e \in N$. (Put briefly: idempotents modulo nilpotents lift.)

(c) Show that if e is a non-trivial idempotent in R, then $V(e)$ and $V(1-e)$ are proper disjoint closed sets in $\mathrm{Spec}\,(R)$ whose union is all of R. That is, $\mathrm{Spec}\,(R)$ is not connected. Show conversely that if $\mathrm{Spec}\,R$ is the union of two disjoint closed proper subsets then they have the form $V(e)$ and $V(1-e)$ for some non-trivial idempotent e in R. (Thus, R is a product in a non-trivial way $\Leftrightarrow$ R has a non-trivial idempotent $\Leftrightarrow$ $\mathrm{Spec}\,R$ is not connected.)