Math 614, Fall 2008 **Supplementary Problem Set #3**

1. Let K be a field and $S = K[x, y]$ be the polynomial ring in two variables over K. For every real number $\beta \geq 0$ let $R_\beta = K[x^m y^n : n > \beta m] \subseteq S$. Show that R_β is not Noetherian and, therefore, not finitely generated as a K-algebra. This gives an uncountable chain of such subrings inside $K[x, y]$, for if $\beta < \gamma$, then $R_\beta \supset R_\gamma$.

2. Let $R = \mathbb{C}[xu, xv, yu, yv] \subseteq \mathbb{C}[x, y, u, v]$, where x, y, u, v are indeterminates. Represent R explicitly as a finite module over a polynomial subring A over $\mathbb{C}$. Give the algebraically independent generators of A as a $\mathbb{C}$-algebra, and give a finite set of generators for R as an A-module. (You may want to begin by finding an algebraic relation on xu, xv, yu, yv.)

3. Let K be an infinite field and let $f \neq 0$ be in $R = K[x_1, \ldots, x_n]$. Show that for suitable $\lambda_i \in K, \lambda_n \neq 0$, there is a K-linear automorphism ϕ of R that sends x_i to $x_i + \lambda_i x_n$, $1 \leq i < n$, and x_n to $\lambda_n x_n$ such that $\phi(f)$ is a nonzero scalar multiple of a polynomial that is monic in x_n with coefficients in $K[x_1, \ldots, x_{n-1}]$. (This leads to an alternative proof of Noether normalization for infinite K in which the algebraically independent elements are K-linear combinations of the original generators.)

4. (a) Consider a chain of ideals in a ring R each of which is not finitely generated. Show that the union I of the chain is also not finitely generated.

(b) It follows from Zorn's lemma that if R is not Noetherian, there is an ideal P that is maximal with respect to not being finitely generated. Prove that such an ideal P is prime. (Thus: R is Noetherian iff every prime ideal is finitely generated (Cohen).)

5. Let K be an algebraically closed field and L an extension field.

(a) Given that a certain finite system of polynomial equations in finitely many variables with coefficients in K has a solution in L, prove that it has a solution in K.

(b) Consider a polynomial $f \in K[x_1, \ldots, x_n] - \{0\}$. Show that if f is the product of two polynomials of strictly smaller degree in $L[x_1, \ldots, x_n]$, then this is also true in $K[x_1, \ldots, x_n]$. (Perhaps part (a) will be helpful.)

6. (E. Noether) Let $A \subseteq R$ be rings with A Noetherian and R finitely generated over A. Let $G = \{g_1, \ldots, g_n\}$ be a finite group acting on R by A-algebra automorphisms. Show that the fixed ring $R^G = \{r \in R : \text{for all } g \in G,\, g(r) = r\}$ is finitely generated over A. (Let $R = A[r_1, \ldots, r_k]$. For each j, show that the elementary symmetric functions e_{ij} of $g_1(r_j), \ldots, g_n(r_j)$ are in R^G. Let $R_0 = A[e_{ij} : 1 \leq i \leq n, 1 \leq j \leq k]$. Show that R is integral and finitely generated over R_0, and use this and that $R_0 \subseteq R^G \subseteq R$.)

EXTRA CREDIT Let $R \subseteq S$ be rings and s in S be such that for every prime Q of S, the image of s in S/Q is integral over $R/(Q \cap R)$. Show that S is integral over R. (This is an alternative way of doing problem **5.** of Problem Set #2.)