Math 614, Fall 2008 **Supplementary Problem Set #4**

1. Let $\{S_i : i \in I\}$ be a possibly infinite family of normal domains, all of which are subrings of the integral domain S. Prove that $\bigcap_{i \in I} S_i$ is normal.

2. Is the ring of continuous real-valued functions on the closed unit interval a Noetherian ring? Prove your answer.

3. (a) Show: if $f : X' \to X$ is continuous with X' irreducible then $f(X')$ is irreducible.

(b) Let K be an algebraically closed field, and let $0 < t \leq r \leq s \in \mathbb{N}$. Identify the set of $r \times s$ matrices over K (they correspond to K-linear maps $K^s \to K^r$) with $K^{rs} = \mathbb{A}_K^{rs}$. Then X_t, the subset of matrices of rank at most t, is a closed algebraic set, defined by the vanishing of size $t+1$ minors.

(1) Show that an $r \times s$ matrix over K has rank $\leq t$ iff it is the product of an $r \times t$ matrix and a $t \times s$ matrix over K. (Suggestion: rank $M \leq t$ iff $\operatorname{Im} M \subseteq$ some t-dimensional subspace of K^r.)

(2) Show that there is a morphism of $\mathbb{A}_K^{rt+ts} = \mathbb{A}_K^{rt} \times \mathbb{A}_K^{ts}$ onto X_t.

(3) Show that X_t is irreducible. What does this imply about the ideal generated, in $K[x_{ij} : 1 \leq i \leq r, 1 \leq j \leq s]$, by the size t minors of an $r \times s$ matrix $\big(x_{ij}\big)$ with indeterminate entries?

4. Let $R \subseteq S$ be rings and suppose that there is an R-linear map $\phi : S \to R$ such that $\phi(r) = r$ for all $r \in R$. (We showed in class that this implies $IS \cap R = I$ for all I in R.) We shall say that R is a *direct summand* of S in this situation.

(a) Show that if S is Noetherian then so is R.

(b) Show that $R[[x_1, \ldots, x_n]]$ is a direct summand of $S[[x_1, \ldots, x_n]]$.

(c) Let R_i be an increasing sequence of subrings of the Noetherian ring S such that every R_i is a direct summand of S. Prove that the union $\bigcup_{i=1}^{\infty} R_i$ is Noetherian.

(d) Let K_i be an increasing sequence of subfields of a field L. Show that the ring $\bigcup_{i=1}^{\infty} K_i[[x_1, \ldots, x_n]]$ is Noetherian.

5. Let $R = K[x, y]$ be a polynomial ring in two variables, and let $m = (x, y)R$. Show that the element $\epsilon = x \otimes y - y \otimes x$ in $m \otimes_R m$ is not 0 but is killed by m, and show that it spans the kernel of the surjection $m \otimes_R m \twoheadrightarrow m^2$ induced by the bilinear map $m \times m \to m^2$ sending (u, v) to uv.

6. Let $K \subseteq L$ be fields, K algebraically closed, and let D be a domain that is a K-algebra. Prove that $L \otimes_K D$ is a domain. (Suggestion: Let b_i be a K-basis for D over K, which will also give an L-basis for $L \otimes_K D$. For all i, j one has, uniquely, $b_i b_j = \sum_k c_{ijk} b_k$ with $c_{ijk} \in K$: all but finitely many c_{ijk} are 0. Think of the problem of finding two nonzero elements u, v of $L \otimes_K D$ whose product is 0 as an equational problem, using indeterminate coefficients to write u, v in terms of specific b_i. The condition that a coefficient $U_i \neq 0$ can be expressed by an auxiliary equation $U_i Z = 1$. **5.**(a) of Problem Set #3 may be helpful.)