Math 614, Fall 2008 **Supplementary Problem Set #5**

1. Let R be a ring and M be an R-module with a finite set of generators $u_1, \ldots, u_n$. Suppose that the module of relations on the u_i is finitely generated, i.e., the module $\{(r_1, \ldots, r_n) \in R^n : r_1u_1 + \cdots + r_nu_n = 0\}$ is finitely generated. (Thus, M is finitely presented.) Show that the module of relations on any other finite set of generators is also finitely generated. [Suggestion: You might want to compare each set of generators with the union, and reduce to the case where the two sets differ in one redundant generator.]

2. Let $f : M \to M$ be a surjective R-linear map of a Noetherian R-module onto itself. Prove that f is injective.

3. Let G be a group acting by ring automorphisms on the domain R.

(a) Show that the action of G on R extends to an action on the fraction field F of R.

(b) Show that if R is normal, then $R^G = \{r \in R : g(r) = r \text{ for all } g \in G\}$ is normal. [There is a *very* short proof.]

(c) Show that if G is finite, then the fixed ring F^G is the fraction field of R^G.

(d) Let K be an infinite field, let $R = K[x, y]$, a polynomial ring, and let G be $K - \{0\}$ under multiplication. Let $c \in K - \{0\}$ act on R by sending $f(x, y)$ to $f(cx, cy)$. Show that F^G is strictly larger than the fraction field of R^G.

4. Let R be a ring graded by $\mathbb{N}$, i.e., R has an abelian group direct sum decomposition $R = \bigoplus_{i=0}^{\infty} R_i$ such that $1 \in R_0$ and for all $i, j \in \mathbb{N}$, if $r \in R_i$ and $r' \in R_j$ then $rr' \in R_{i+j}$. Note that R_0 is a subring and that $J = \bigoplus_{i=1}^{\infty} R_i$ is an ideal. Elements of R_i are called *homogeneous of degree* i. Show that the following conditions are equivalent:

(a) R is a Noetherian ring.

(b) R_0 is Noetherian ring and J is a finitely generated ideal.

(c) R_0 is a Noetherian ring and R is finitely generated as an R_0-algebra.

[Suggestion: For (b) $\Rightarrow$ (c), show that if (b) holds then J has a finite set of homogeneous generators as an ideal, and these generate R as an R_0-algebra.]

5. Find an irredundant primary decomposition of $(xy, xz, yz^2, z^3)R$ in the polynomial ring $K[x, y, z]$ over the field K. What are the minimal primes? Are there any embedded primes, and if so, what are they? Which primary ideals in the decomposition are unique and which are not? Justify your answers, including why the ideals you use are primary.

6. Let (R, m) be a Noetherian local ring and let $x \in m$ be an element that is not a zerodivisor. Suppose that R/xR is a domain. Prove that R is a domain.

EXTRA CREDIT Show that **2.** above holds under the weaker hypothesis that M be finitely generated.

EXTRA CREDIT (a) Show that if $x, y \in X = \mathrm{Spec}\,(R)$, then either x and y have disjoint open neighborhoods or x and y are both in the closure of some point $z \in X$.

(b) Let Y be the space of all minimal primes of $X = \mathrm{Spec}\,(R)$. Show that for all $a \in R$, $D(a) \cap Y$ is both open and closed (i.e., *clopen*) in Y, so that Y is T_2 with a basis of clopen sets. Here, $D(a) = X - V(a)$.