Math 614, Fall 2010 **Problem Set #1**

Due: Friday, October 1

Throughout, R is a commutative ring with identity, K is a field and $K[x]$, $K[x,\, y]$ are polynomial rings over K. If M is an R-module, $M^* = \mathrm{Hom}_R(M,\, R)$.

1. (a). Let $f \in K[x] = R$ have degree $d \geq 1$. You may assume that f is transcendental over K, so that $A = K[f] \subseteq K[x]$ is isomorphic to a polynomial ring in one variable over K. Prove that every element of R is uniquely expressible in the form $a_0 + a_1 x + \cdots + a_{d-1}x^{d-1}$, where every $a_j \in A$. In particular, $R \cong A^d$ as an A-module.

(b) Let B denote any subring of R containing K, and let I be an ideal of B. Show that B is finitely generated as a K-algebra. Show also that if B contains a polynomial of degree $d \geq 1$, then every ideal of B can be generated by d or fewer elements.

2. (a) Let h be a positive integer. Prove that x^{9h}, $x^{6h+1} + x^{3h+2}$, and $x^{3h} + x$ generate $K[x]$ as a ring over K.

(b) Consider the subring $T = K[x^3,\, x^2 + x] \subseteq K[x] = R$. Show that R/T is a nonzero finite-dimensional K-vector space, and find a basis for it over K. In particular, $T \neq R$.

(c) Show that x *is* in the field of fractions of T.

3. Let $R = K[x,\, y]$. Let a be a positive real number. Let $S_a \subseteq R$ be the subring of R spanned as a K-vector space by all monomials $x^m y^n$, $m,\, n \in \mathbb{N}$, such that $m > an$. Show that if $0 < a < b$, then $S_b \subset S_a$, and this containment is strict. Show that for all $a > 0$, S_a contains an infinite strictly increasing chain of ideals.

4. Let R be an integral domain with fraction field $\mathcal{F}$, and let $(0) \neq I \subseteq R$ be an ideal.

(a) Let $H = \{f \in \mathcal{F} : fI \subseteq R\}$. Show that $H \cong \mathrm{Hom}_R(I,\, R) = I^*$.

(b) Let $R = K[x^2,\, x^3] \subseteq K[x]$. Let I be the ideal $(x^2,\, x^3)R \subseteq R$. Is the natural map $I \to I^{**}$ an isomorphism?

(c) Let R be a unique factorization domain and let I be an ideal of R. Suppose that I contains relatively prime elements $u,\, v$. Prove that $I^* \cong R$ and, hence, $I^{**} \cong R$.

5. Let P and Q be prime ideals of R. Show that P and Q have disjoint open neighborhoods if and only if there is no prime ideal P_0 that is contained in both P and Q. ("Only if" is the easier part. Note that $\{ab : A \in R \backslash P, b \in R \backslash Q\}$ is a multiplicative system in R.)

6. Let $\mathcal{C}$ be a category, and X, Y be objects of $\mathcal{C}$. A morphism $f : X \to Y$ is called an *epimorphism* if for all objects Z and $g, h : Y \to Z$, whenever $g \circ f = h \circ f$, then $g = h$. (In the categories of sets or modules over a ring R, an epimorphism can be shown to be the same thing as a surjective morphism. In the category of Hausforff spaces, if $f : X \to Y$ and $f(X)$ is dense in Y, then f is an epimorphism.) Let $f : R \to S$ be a ring homomorphism, and suppose that every element of S has the form $f(r)/f(u)$, where $r,\, u \in R$ and $f(u)$ is invertible in S. (E.g., one may take $R = \mathbb{Z}$ and $S = \mathbb{Z}[1/2] = \{r/2^h : r \in \mathbb{Z},\, h \in \mathbb{N}\}$.) Prove that f is an epimorphism in the category of rings.

EXTRA CREDIT 1 Suppose that z is an indeterminate over R and that $R[z] \cong K[x,\, y]$. Prove that R is isomorphic to the polynomial ring in one variable over K.