Math 614, Fall 2010 **Problem Set #2**
Due: Friday, October 15

1. Let X, Y be nonempty topological spaces.

(a) Prove that X is irreducible iff every nonempty open set is dense in X.

(b) If X and Y are irreducible, is $X \times Y$ necessarily irreducible? Prove your answer.

2. Let R be a ring and let e be an idempotent element of R, so that $e^2 = e$. Let $f = 1 - e$, *the complementary idempotent* to e.

(a) Show that the ideal Re is a ring with identify e. Note that it is not a subring of R, unless $e = 1$. (Likewise, Rf is a ring with identity f.)

(b) Show that $R \cong Re \times Rf$ (the product ring) via the maps $r \mapsto (re, rf)$ and $(u, v) \mapsto u + v$.

Note that the product $R = S \times T$ arises, up to ismorphism, in this way from the idempotents $(1_S, 0)$ and $(0, 1_T)$. That is, up to isomorphism, giving a product decomposition of R into two factors is equivalent to giving an idempotent element of R.

Extra credit 2 If $h \in R$ is such that $h - h^2$ is nilpotent, show that $h = e + n$ where e is idempotent and n is nilpotent, and that e and n are unique.

3. Let $R \subseteq S$ be rings and let $\theta : S \to R$ be an R-module map that is a splitting, so that for $r \in R$, $\theta(r) = r$.

(a) Show that for every ideal I of R, $IS \cap R = I$.

(b) Prove that if S is a normal domain, then R is a normal domain.

(c) Let $S = K[x, y, u, v]$ be the polynomial ring in four variables over a field K. Let $R = K[xu, xv, yu, yv] \subseteq S$. Show that $R \subseteq S$ has an R-module splitting.

4. Let $R \subseteq S$ be integral domains and let W be a multiplicative system in R that does not contain 0. Thus, $W^{-1}R \subseteq W^{-1}S$. Suppose that $s \in S$ and $w \in W$. Prove that s/w is integral over $W^{-1}R$ if and only if for some $v \in W$, vs is integral over R. (Hence, if T is the integral closure of R in S, then $W^{-1}T$ is the integral closure of $W^{-1}R$ in $W^{-1}S$.)

5. Let X be the space of minimal prime ideals of the commutative ring R in the inherited Zariski topology. Prove that X is a Hausdorff space. Must X be totally disconnected? (This means that every connected component of X consists of just one point.)

6. A topological space X is called *Noetherian* if every infinite descending chain of closed sets is eventually constant.

(a) Prove that every subspace of a Noetherian space is Noetherian, and that a Noetherian space is quasicompact.

(b) Prove that X is Noetherian iff every open subspace is quasicompact.

Extra credit 3 Prove that if X is Noetherian, every nonempty closed subset is a finite union of irreducible closed sets.