Math 614, Fall 2008 **Problem Set #3**
Due: Friday, November 5

1. (a) Let M be a module that is not Noetherian. Show that among the submodules of R that are not finitely generated, there is a maximal one.

(b) Let R be a ring that is not Noetherian, and let I be maximal among ideals that are not finitely generated. Show that I is prime. (If $ab \in I$, $a \notin I$, $b \notin I$, one may consider $I + aR$ and $J = \{r \in R : ra \in I\}$.)

2. Let R be any commutative ring, and let $S = R[x_1, \ldots, x_n]$ be the polynomial ring in n variables over R. Let $Q_0 \subset Q_1 \subset \cdots \subset Q_k$ be a strictly ascending chain of prime ideals of S all of which lie over the same prime P in R. Show that $k \leq n$.

3. Let $K \subseteq L$ be fields with K algebraically closed. Let T, V, and W be L-vector spaces with respective bases $\{t_h : h \in H\}$, $\{v_i : i \in I\}$, and $\{z_j : j \in J\}$. Let $B : T \times V \to W$ be an L-bilinear map. Let T_K, V_K, and W_K be the K-spans of the respective bases. Suppose that B maps $T_K \times V_K$ into W_K. Prove that if there are nonzero vectors $t \in T$, $v \in V$ such that $B(t, v) = 0$, then one can choose nonzero $t' \in T_K$ and $v' \in V_K$ so that $B(t', v') = 0$.

4. Let R be a ring, and let G be a group acting on R by ring automorphisms. Let $R^G = \{r \in R : g(r) = r \text{ for all } g \in G\}$, the *ring of invariants*. Suppose also that $G = \{g_1, \ldots, g_n\}$ is finite, where $g_1 = e$ is the identity.

(a) Show that for all $r \in R$, $(X - g_1(r))(X - g_2(r)) \cdots (X - g_n(r)) \in R^G[X]$. Since $g_1(r) = r$, conclude that R is an integral extension of R^G.

(b) Let $A \subseteq R^G$, and suppose that $r_1, \ldots, r_m$ generate R as an A-algebra. Prove that there is a finitely generated A-algebra $S \subseteq R^G$ such that R is module-finite over S. Conclude that if A is Noetherian, then R^G is fintiely generated over A. (E. Noether)

(c) Suppose that n $(= n \cdot 1_R)$ is invertible in R. Show that $\rho : R \to R^G$ defined by $\rho(r) = (1/n) \sum_{j=1}^{n} g_j(r)$ is an R^G-module retraction $R \to R^G$.

5. Let $R = R_0 \oplus R_1 \oplus \cdots \oplus R_n \oplus \cdots$ be an $\mathbb{N}$-graded algebra over a Noetherian ring R_0. Let I be the ideal $\bigoplus_{n=1}^{\infty} R_n$. Prove that R is Noetherian iff the ideal I is generated by finitely many homogeneous elements $F_1, \ldots, F_m$, in which case $R = R_0[F_1, \ldots, F_m]$.

6. Let $R \hookrightarrow S$ be a ring extension that is split as a map of R-modules.

(a) Show that if S is Noetherian (respectively, Artinian), then so is R.

(b) Let S be $\mathbb{N}$-graded and $k > 0$ be an integer. Let $R = \bigoplus_{n=0}^{\infty} S_{nk}$. Show that $R \hookrightarrow S$ splits over R. (R is called the kth *Veronese subring* of S.)

(c) Show that the Veronese subrings of the polynomial ring $K[x_1, \ldots, x_n]$ (graded as usual) are normal, and finitely generated over K. (Suggestion: use previous problems.)

Extra Credit 4 Let $R \subseteq S$ and suppose that $u \in S$ is such that its image in S/Q is integral over $R/(Q \cap R)$ for all $Q \in \operatorname{Spec}(S)$. Must u be integral over R?

Extra Credit 5 Let K be a field and $x_1, \ldots, x_r$ and $y_1, \ldots, y_s$ indeterminates over K. Show that $K[x_i y_j : 1 \leq i \leq r, 1 \leq j \leq s] \subseteq K[x_1, \ldots, x_r, y_1, \ldots, y_s]$ is normal.