Math 614, Fall 2010 **Problem Set #4**

Due: Friday, November 19

1. Let u and v be relatively prime elements in the UFD R, and let W be an R-module. Let $J = (u,\, v)R$, assume $J \neq R$, and let $f : R^2 \twoheadrightarrow J$ be such that $f(r_1, r_2) = r_1 u + r_2 v$.

(a) Show that the kernel of f is the cyclic module generated by $(v,\, -u)$.

(b) Show that $W \otimes_R J \cong (W \oplus W)/\{(vw,\, -uw) : w \in W\}$.

(c) Show that $W \otimes_R J \to W \otimes_R R \cong W$ is injective if and only if whenver $w_1, w_2 \in W$ are such that $uw_1 = vw_2$, there exists $w \in W$ such that $w_1 = vw$ and $w_2 = uw$.

(d) Prove that $J \otimes_R J \to J$ with $j \otimes j' \mapsto jj'$ is not injective. Hence, J is not R-flat.

2. Let I be a directed partially ordered set, and let $\{M_i\}_{i \in I}$ be a direct limit system of R-modules. Let $M = \varinjlim_i M_i$ be the direct limit.

(a) Fix $i \in I$. Show that the modules $N_{ij} = \mathrm{Ker}\,(M_i \to M_j)$ form a directed union indexed by $\{j \in I : j \geq i\}$.

(b) Let $N_i = \bigcup_{j \geq i} N_{ij}$. Show that $N_i = \mathrm{Ker}\,(M_i \to M)$ for all $i \in I$.

(c) Show that the modules $\overline{M}_i = M_i/N_i$ form a direct limit system indexed by I with maps induced by the maps $M_i \to M_j$, and that the map $\overline{M}_i \to \overline{M}_j$ is injective for $j \geq i$. Hence, M is the directed union of the isomorphic images of the $\overline{M}_i$ in M.

3. Let M be a Noetherian R-module and let let $f : M \to M$ be surjective. Prove that f is an isomorphism.

Extra Credit 6 Let M be a finitely generated R-module and let let $f : M \to M$ be surjective. Prove that f is an isomorphism.

4. Let R be a principal ideal domain. Let $a,\, b \in R - \{0\}$ with $\mathrm{GCD}(a,\, b) = d$.

(a) Show that $(R/aR) \otimes_R (R/bR) \cong R/dR$.

(b) Show also that $\mathrm{Hom}_R(R/aR,\, R/bR) \cong R/dR$.

(c) Prove that for any two finitely generated torsion modules $M,\, N$ over R, $M \otimes_R N \cong \mathrm{Hom}_R(M,\, N)$. (This isomorphism is *not* natural.)

5. Let R be a reduced ring of Krull dimension 0. Prove that for every prime ideal P of R, R_P is a field, and that every R-module is flat.

6. Recall that in any category $\mathcal{C}$, a morphism $f : X \to Y$ is an *epimorphism* if whenever $g,\, h : Y \to Z$ and $g \circ f = h \circ f$, then $g = h$. Let $T,\, S$ be R-algebras.

(a) Show that $R \to S$ is an epimorphism of rings if and only if the map $S \otimes_R S \to S$ induced by the R-bilinear map $S \times S \to S$ such that $(s_1,\, s_2) \mapsto s_1 s_2$ is an isomorphism.

(b) Prove that if R is a field, an epimorphism $R \to S$ is an isomorphism or else $S = 0$.

(c) Prove that if $R \to S$ is an epimorphism, then so is $T = T \otimes_R R \to T \otimes_R S$.

(d) Prove that if $R \to S$ is an epimorphism, P is prime in R, and $\kappa_P = R_P/PR_P$, then $\kappa_P \otimes_R S$ (the scheme-theoretic fiber), as a κ_P-algebra, is κ_P or is 0.