Math 614, Fall 2010 **Problem Set #5**
Due: Monday, December 13

1. Let M and N be finitely generated modules over the Noetherian ring R. Let $I = \mathrm{Ann}_R M$ and let $J = \mathrm{Ann}_R N$. Show that the support of $M \otimes_R N$ is $V(I+J)$ (there is a class result that is relevant). Show that the support of $\mathrm{Hom}_R(M,\, N)$ is $\subseteq V(I+J)$.

2. Let $I \neq R$ be an ideal of a Noetherian ring R and let $I = Q_1 \cap \cdots \cap Q_n$ be an irredundant primary decomposition of I. Let $P_i = \mathrm{Rad}\,(Q_i)$. For all $N \gg 0$, let $Q_{i,N}$ be the contraction to R of $(P_i^N + I)R_{P},$. Show that for all $N \gg 0$, $Q_{i,N} \subseteq Q_i$, is primary to P_i, and that replacing Q_i by $Q_{i,N}$ for such an $N \gg 0$ gives a primary decomposition of I.

3. Let P be a prime ideal of R, a Noetherian ring, and let $W = R - P$. Let $P^{(n)}$ be the nth sybolic power of P, i.e., the contraction of $P^n R_P$ to R. Let J be $\bigcup_{w \in W} \mathrm{Ann}_R w$. Prove that $J = \bigcap_{n=1}^{\infty} P^{(n)}$.

4. Let R be a Noetherian ring, and let M be a finitely generated R-module. Let I be an ideal of R. Let $N = \mathrm{Ann}_M I$. Prove that $\mathrm{Ass}\,(M) = \mathrm{Ass}\,(N) \cup \mathrm{Ass}\,(M/N)$. (By a class theorem, one has $\subseteq$. The problem is to prove $\supseteq$, which is false in general but true here.)

5. Let R be a Noetherian graded ring over $\mathbb{N}^h$ or $\mathbb{Z}^h$, where $h \geq 1$, and let M be a graded module (for $\mathbb{N}^h$ or $\mathbb{Z}^h$)

(a) Show that all associated primes of M are homogeneous ideals (for the $\mathbb{N}^h$ or $\mathbb{Z}^h$ grading). [Suggestion: if P is the annihilator of a nonzero element $u \in M$, replace u by a nonzero multiple $v = v_1 + \cdots + v_d$, where the v_i are the nonzero homogeneous components of v, such that d is as small as possible. Then show that all of the v_i have annihilator P.]

(b) Let R be a polynomial ring $K[x_1,\, \ldots\,, x_n]$ over a field K, and give R the $\mathbb{N}^n$-grading in which the $(a_1,\, \ldots\,, a_n)$-forms are the elements of the one-dimensional K-vector space $Kx_1^{a_1} \cdots x_n^{a_n}$. Let I be a proper ideal of R generated by monomials. Prove that every associated prime of I is generated by a subset of the variables, and that I has a primary decomposition in which every ideal is generated by monomials.

6. Find a primary decomposition of $(wxyz^2,\, x^2,\, y3,\, xyz)R$ in the polynomial ring $K[w,\, x,\, y,\, z]$ over a field K. Which associated primes are minimal and which are embedded? Which primary components in your decomposition are unique?

Extra Credit Let R be a ring whose localizations at maximal ideals are all Noetherian.

(a) Show that if every element of R is contained in only finitely many maximal ideals, then R is Noetherian.

(b) Give an example where R is not Noetherian.

(c) Let $R = K[x_1,\, \ldots\,, x_n, \ldots]$ be the polynomial ring in an infinite sequence of variables over a field K. Partition the variables into sets $S_1,\, \ldots\,, S_n,\, \ldots$ such that S_n contains n of the variables. Let P_n be the prime ideal generated by S_n. Let $W = S - \bigcup_{n=1}^{\infty} P_n$. Prove that $W^{-1}S_n$ is Noetherian but has infinite Krull dimenson.