Math 614, Fall 2010 **Additional Problems**

1. Let P be a prime ideal of a polynomial ring $R = K[x_1, \ldots, x_n]$ over a field K. Let L be a field extension of K. Prove that every minimal prime of $PL[x_1, \ldots, x_n]$ has the same height as P.

2. Let R be as Problem **1.** Let P and P' be prime ideals in R, and let Q be a minimal prime of $P+Q$. Prove that the height of Q is at most the sum of the heights of P and P'.

3. Let D be a Dedekind domain.

(a) Suppose that D has only finitely many nonzero prime ideals, $P_1, \ldots, P_n$. Show that for every P_i there is an element in P_i not in $P_i^{(2)}$ nor in any P_j for $j \neq i$. Prove that this element generates P_i.

(b) Prove that if D has only finitely many nonzero primes, then D is a PID.

(c) Let $f \in D$ be nonzero, and let $P_1, \ldots, P_n$ be the primes that contain f. Let W be the multiplicative system $D - \bigcup_{j=1}^{n} P_j$. Prove that $W^{-1}D/(f) \cong D/fD$. Conclude that every ideal in D/fD is principal.

(d) Prove that every ideal I of D is generated by two elements, one of which may be chosen arbitrarily to be any nonzero element of I.

4. Let R be a domain finitely generated over a field of characteristic 0. Let $\mathcal{L}$ be a finite algebraic extension of the fraction field of R. Prove that the integral closure of R in $\mathcal{L}$ is module-finite over R.

5. Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring over a field and let $B \subseteq R$ be generated by finitely many monomials. Prove the B is normal if and only if for every monomial μ that is the ratio of two monomials in B, if $\mu^k \in B$ for some positive integer k, then $\mu \in B$.

6. Let R be a normal Noetherian domain. Let $I \subseteq R$ be a nonzero proper ideal. Prove that I is a reflexive R-module if and only if every associated prime of I has height one.