Math 614, Fall 2008 **Supplementary Problem Set #2**

1. Let $f : R \to S$ be a ring homomorphism, V a multiplicative system in R, and W the image of V in S.

(a) Explain carefully why there is a unique induced ring homomorphism $g : V^{-1}R \to W^{-1}S$ such that $g(r/1) = f(r)/1$ for all $r \in R$.

(b) Show that if S is module-finite over R (respectively, integral), then $W^{-1}S$ is module-finite (respectively, integral) over $V^{-1}R$.

2. Suppose in **1.** (a) that $R \subseteq S$ is a subring (and then $W = V$). Let T be the integral closure of R in S. Show that $V^{-1}R \to V^{-1}S$ is injective, and that the integral closure of its image in $V^{-1}S$ is $V^{-1}T$.

3. (a) Which elements in the polynomial ring $K[x,\, y,\, z]$ over the field K are integral over $K[x^7,\, y^{11},\, z^{13}]$? Explain your answer.

(b) Let S be the ring of elements in $\mathbb{Q}[\sqrt{11}]$ integral over $\mathbb{Z}$. Show that there is an element $s \in S$ such that $S = \mathbb{Z} + \mathbb{Z}s$. Give s explicitly.

4. Let $A \subseteq S$ be rings and let $f,\, g \in S[x]$ be monic polynomials. Let R be the ring generated over A by the coefficients of the product polynomial fg. Show that if S is a domain, then every coefficient of f and of g is integral over R. [Suggestion: Enlarge S to an algebraically closed field L. Explain why all the roots of fg are integral over R. Express the coefficients of f and of g in terms of these roots.]

5. Prove the statement in problem **4.** without the assumption that S is a domain.

6. Suppose that R is a principal ideal domain and that $R[z]$, the polynomial ring in one variable over R, is isomorphic to $S = K[x,\, y]$, the polynomial ring in two variables over a field K. Prove that $R \cong K[u]$, a polynomial ring in one variable over K. [Identify R and z with their images in S. Let u be a generator of $m \cap R$, where $m = (x,\, y)S$. One approach is to prove that every $G \in R$ is in $K[u]$ by induction on the degree of G considered as a polynomial in S.]