Math 614, Fall 2012 **Problem Set #1**

Due: Wednesday, September 26

1. Let R be a commutative ring and x an indeterminate over R.

(a) Show that an element of $R[x]$ is nilpotent if and only if all of its coefficients are nilpotent.

(b) Characterize the elements of $R[x]$ that are invertible.

2. Let R be the ring of all functions f on the set P of positive prime integers $\{2, 3, 5, 7, \dots\}$ such that (1) for every $p \in P$, $f(p) \in \mathbb{Z}/p\mathbb{Z}$, the field with p elements, and (2) there exists an integer n_f and $q_f \in P$ such that for all $p \in P$ with $p \geq q_f$ $f(p) \equiv n_f \bmod p$. Show that the set M of minimal prime ideals of R is countably infinite, and contains a unique element Q that is not maximal. Show that M, in the inherited Zariski topology, is homeomorphic to a compact subset of $\mathbb{R}$, that there are only countably many prime ideals of R that strictly contain Q, and that they are all maximal.

3. Let X be a set and let R be the set of all subsets of X. R is a commutative ring in which the sum of $A, B \subseteq X$ is $A \triangle B := (A \backslash B) \cup (B \backslash A)$ and the product is $A \cap B$. You may assume this. What is the 0 element for addition? What is the additive inverse of A? What is the identity for multiplication? Show that a family of subsets of X is an ideal of R iff it is closed under taking subsets and finite union. Show that if $x \in X$, then $m_x = \{Y \subseteq X : x \notin Y\}$ is a maximal ideal of R. Show that if X is infinite, then R has a maximal ideal different from all of the ideals m_x. (Note that the set of finite subsets of X is a proper ideal.) What fields can be obtained as R/m for m a maximal ideal of R?

4. (a) Let $R \subseteq S$ be rings and $P \in \operatorname{Spec} R$. Show that there exists a prime Q of S whose contraction to R is P if and only if the map $R \to R/P$ extends to a map $S \to D$, where $R/P \subseteq D$ and D is an integral domain.

(b) Let $S = K[x, y, z]$ be the polynomial ring in three variables, and $R = K[xy, yz, zx] \subseteq S$. Is $\operatorname{Spec}(S) \to \operatorname{Spec}(R)$ surjective? If not, give an explicit prime not in the image, and describe the image, if possible, as the union of an open set and a closed set.

5. Let $R = K[x, y, z]$ be a polynomial ring in x, y, z over a field K and let M be the R-module $I \oplus (R^2/Rv)$, where I is the ideal $yR + zR$, and $v = (x, y) \in R^2$. Show that M is torsion-free, and give finitely many explicit R-module generators for $\operatorname{Hom}_R(M, R)$. Give as simple and explicit a description of $\operatorname{Hom}_R(M, R)$ as you can.

6. An object X of a category C is called an *initial object* (resp., a *terminal object*) if for all objects Y of C, $\operatorname{Mor}(X, Y)$ (resp., $\operatorname{Mor}(Y, X)$) has precisely one element. For each of the following categories, decide whether there is an initial object and whether there is a terminal object. If one or both of these exists, give an explicit description.

(a) Sets and functions.

(b) Commutative rings with identity and ring homomorphisms that preserve the identity.

(c) Commutative rings not necessarily with identity and all ring homomorphisms.

(d) Modules over a commutative ring R and R-module homomorphisms.

EXTRA CREDIT 1 Let u be an element of a ring R such that $u^2 - u$ is nilpotent. Show that there is a nilpotent element n in R such that $(u + n)^2 - (u + n) = 0$.

EXTRA CREDIT 2 Let $R \subseteq S$ be rings and let $s \in S$. Suppose that for every minimal prime Q of S, there is a monic polynomial f_Q in the polynomial ring $R[x]$ such that $f_Q(s) \in Q$. Show that there is a monic polynomial $f \in R[x]$ such that $f(s) = 0$.