Math 614, Fall 2012 **Problem Set #2**

Due: Friday, October 12

1. (a) Let $X = \mathrm{Spec}\,(R)$ and let $\{D(f_\lambda) : \lambda \in \Lambda\}$ be a family of open sets such that any finite number of elements in the family have nonempty intersection. Show that the intersection of the family is nonempty.

(b) Let $h : R \to S$ be a ring homomorphism. Show that under $\mathrm{Spec}\,(h)$, inverse images of quasicompact open sets are quasicompact open sets.

2. Let R be a ring of prime characteristic $p > 0$. Let F denote the homomorphism of $R \to R$ such that $F(r) = r^p$. Prove that $\mathrm{Spec}\,(F)$ is the identity map on $\mathrm{Spec}\,(R)$.

3. Let $a,\, m,\, b \neq 0,\, 1$ and $d > 0$ be integers such that b is square-free and $\mathrm{GCD}(a,\, m,\, d) = 1$. Characterize the values of $a,\, m,\, b,\, d$ such that $(a + m\sqrt{b})/d$ is integral over $\mathbb{Z}$.

4. Let M be a faithful R-module generated by n elements. Suppose that $I \subseteq J$ are ideals of R such that $I \subseteq J$ and $JM = IM$. Show that every element $j \in J$ satisfies a monic polynomial $x^n + i_1x^{n-1} + \cdots + i_tx^{n-t} + \cdots + i_{n-1}x + i_n = 0$ where for $1 \leq t \leq n$, $i_t \in I^t$.

5. Let $S = R[x]$, the polynomial ring in one variable over R. Show that the a chain of prime ideals of S lying over a given prime ideal P of R has length at most one. Show that if R has finite Krull dimension d, then the Krull dimension n of S is such that $d+1 \leq n \leq 2d+1$. (In the Noetherian case, $n = d+1$. In the general case, the statement made here is sharp.)

6. Let R be a ring and let $S = R[x_1,\, \ldots,\, x_n]$ be the polynomial ring in n variables over R. Let $f_1,\, \ldots,\, f_n \in S$ be such that $f_i = x_i^{d_i} + g_i$ where $d_i \geq 1$ and $g_i \in S$ is such that $\deg(g_i) < d_i$ if $g_i \neq 0$, $1 \leq i \leq n$. Let $T = S/J$, where J is the ideal of S generated by $f_1,\, \ldots,\, f_n$. Show that T is module-finite over R, and give an explicit set of R-module generators for T.

EXTRA CREDIT 3 The *patch* topology (or *strong topology*) on $\mathrm{Spec}\,(R)$ is the topology in which a subbasis for the closed sets consists of subsets of the form $V(I) \cap D(f)$, where $I \subseteq R$ is an ideal and $f \in R$. (The closed sets are the intersections of finite unions of these.) Show that $\mathrm{Spec}\,(R)$ is a compact Hausdorff totally disconnected topological space in this topology, and that if $h : R \to S$ is a ring homomorphism, $\mathrm{Spec}\,(h)$ is continuous for the patch topologies. [Hence, images of closed sets are closed, as well as inverse images.]

EXTRA CREDIT 4 If $\mathcal{C}$ is a category and A is an object, we may consider a category $\mathcal{D}$ whose objects are elements of $\mathrm{Mor}\,(A, X)$ for some X. Given $f : A \to X$ and $g : A \to Y$, a morphism from f to g is a morphism $h : X \to Y$ such that $g = hf$. (If $\mathcal{C}$ is the category of commutative rings and A is fixed ring, the new category constructed in this way is the category of A-algebras.) Suppose that coproducts exist in $\mathcal{D}$. Show that $f : A \to X$ is an *epimorphism* in $\mathcal{C}$ (this means that for all objects Y in $\mathcal{C}$, the map $\mathrm{Mor}\,(X,\, Y) \to \mathrm{Mor}\,(A,\, Y)$ given by composition with f is injective) if and only if the coproduct of f with itself in $\mathcal{D}$, which we denote $X \coprod_A X$, is such that $(\mathrm{id}, \mathrm{id}) : X \coprod_A X \to X$ is an isomorphism.