Math 614, Fall 2012 **Problem Set #3**
Due: Monday, October 29

**1.** Let $R = K[x_1, \ldots, x_n]$ be the polynomial ring in $n$ variables over a field $K$. Let $k$ be an integer with $0 \le k < n$ and let $I_k$ be the ideal generated by all of the products $x_i x_{i+1} \cdots x_{i+k}$, where the subscripts are read modulo $n$. (E.g, if $n = 3$, $I_0 = (x_1, x_2, x_3)R$, while $I_1 = (x_1x_2, x_2x_3, x_3x_1)R$.) What are the minimal primes of $R/I_k$? What is the Krull dimension of this ring?

**2.** Let $R$ be a ring finitely generated over a field $K$, and suppose that $R$ has Krull dimension $d$. Let $S$ be a ring generated over $R$ by $k$ elements. Prove that the Krull dimension of $S$ is at most $d + k$.

**3.** Let $K \subseteq L$ be fields, with $K$ algebraically closed. Let $s$, $d_1, \ldots, d_s$, and $e_1, \ldots, e_s$ be specified positive integers. Let $F \in K[x_1, \ldots, x_n]$, the polynomial ring, and suppose that $(*)$ $F = G_1H_1 + \cdots + G_sH_s$ where $G_1, \ldots, G_s, H_1, \ldots, H_s \in L[x_1, \ldots, x_n]$, and $G_1, \ldots, G_s$ have respective degrees $d_1, \ldots, d_s$ while $H_1, \ldots, H_s$ have respective degrees $e_1, \ldots, e_s$. Show that $G_i$, $H_i$ of these degrees can be chosen in $K[x_1, \ldots, x_n]$ such that $(*)$ holds. (Note that this shows that if $F$ is irreducible in $K[x_1, \ldots, x_n]$, it is irreducible in $L[x_1, \ldots, x_n]$: that is the case where $s = 1$, and $d_1 + e_1$ is $\deg(F)$.)

**4.** Let $A = \big(a_{ij}\big)$ be an $m \times n$ matrix of integers. Let $K$ be a field, and let $x_1, \ldots, x_n$ be indeterminates over $K$, and let $S = K[x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}]$. For $1 \le i \le m$, let $\mu_i = x_1^{a_{i,1}} x_2^{a_{i,2}} \cdots x_n^{a_{i,n}}$. Let $T = K[\mu_1, \ldots, \mu_m] \subseteq S$. Prove that the Krull dimension of $T$ is the rank of the matrix $A$ over $\mathbb{Q}$.

**5.** Let $M$ be a finitely generated module over a commutative ring $R$.
(a) Let $u \neq 0$ be an element that is in every nonzero submodule of $M$. Prove that the annihilator $\{r \in R : ru = 0\}$ is a maximal ideal of $R$.
(b) Let $N$ be a proper submodule of $M$. Prove that $N$ is an intersection (which is allowed to be infinite) of submodules $W_\lambda$ of $M$ with the property that for each $\lambda$ there exists $m_\lambda \in M \backslash W_\lambda$ such that $\{r \in R : rm_\lambda \in W_\lambda\}$ is a maximal ideal of $R$.

**6.** Let $K$ be a field, let $R = K[z_1, \ldots, z_n]$ be a polynomial ring and let $f$ be the sum of the products of the indeterminates taken $n-1$ at a time (so that $f$ has $n$ terms). That is, $f = \sum_{i=1}^{n}(\prod_{j \neq i} z_j)$. Find explicit elements in $S = R/fR$ that are algebraically independent over $K$ and such that $S$ is module-finite over the $K$-algebra they generate.

**Extra Credit 5** Let $R$ be a commutative ring with identity, let $m, n \ge 1$ be integers and let $R^m \hookrightarrow R^n$ be an injective map of free modules. Prove that $m \le n$.

**Extra Credit 6** Let $S$ be any $K$-subalgebra of the polynomial ring $K[x]$ in one variable over a ring $K$. Prove that if $K$ is a field, then $S$ is finitely generated over $K$ and, hence, Noetherian. Is this true when $K = \mathbb{Z}$? Prove your answer.