Math 614, Fall 2012 **Problem Set #4**
Due: Friday, November 16

1. Consider a system of m linear equations in n variables over a commutative ring R, $\sum_{i=1}^{n} r_{ij}x_i = r_j$, where $1 \leq j \leq m$ indexes the equations and the r_{ij}, r_j are given elements of R. Prove that the equations have a solution in R if and only if for every maximal ideal of R the corresponding system in which the coefficients are replaced by their images in R_m has a solution in R_m.

2. Let M be a Noetherian R-module. Let $u \neq 0$ be an element that is in every nonzero submodule. Prove that there is a maximal ideal m of R and an integer $n > 0$ such that $m^n M = 0$. Also show that $\mathrm{Ann}_M m = \{v \in M : mv = 0\}$ is a one-dimensional vector space over $K = R/m$ spanned by u.

3. Let $K \subseteq L$ be fields and let $S = L[[x]]$ be the formal power series ring in one variable over L. Let $R = K + xL[[x]]$, the subring of R consisting of all power series with constant term in K. Prove that R is a Noetherian ring if and only if L is a finite algebraic extension of K. Prove that if L is algebraic over K then that R is normal if and only if $K = L$.

4. Identify the vector space of $r \times s$ matrices over the algebraically closed field K with $\mathbb{A}_K^{rs}$ as discussed in class. Let Z denote the closed algebraic set defined by the vanishing of the ideal generated by $(t+1) \times (t+1)$ minors: Z consists of $r \times s$ matrices of rank at most t. Prove that Z is irreducible. (Suggestion: show that there is a surjection $\mathbb{A}^{rt+ts}$ onto Z, corresponding to the fact that a linear map of rank at most t factors through K^t.) What is the dimension of Z?

5. Let $R = A[x, y]$ be a polynomial ring and let J be the ideal $xR + yR$. Let M be an R-module. Determine the precise conditions on M for the map $J \otimes_R M \to R \otimes_R M \cong M$ to be injective.

6. Let K be an algebraically closed field. Let S denote the set of elements in $K^r \otimes K^s \cong K^{rs}$ that can be written in the form $u \otimes v$ (the set of *decomposable* tensors). Show that S can be regarded as a closed algebraic set in $\mathbb{A}_K^{rs}$.

Extra Credit 7 Let M be a finitely generated R-module and let let $f : M \to M$ be surjective. Prove that f is an isomorphism.

Extra Credit 8 Consider an $n \times n$ matrix $M = (A_{ij})$ whose entries consist of n^2 mutually commuting $m \times m$ matrices over a commutative ring R. Let D be the $m \times m$ matrix obtained by taking the determinant of M. M can also be thought of as a block form for an $mn \times mn$ matrix $\mathcal{M}$ over R. Show that the determinant of the $mn \times mn$ matrix $\mathcal{M}$ is equal to $\det(D)$.