Math 614, Fall 2012 **Problem Set #6**
Due: Monday, December 17

This is an optional assignment. All problems are extra credit problems.

1. (a) Let S be a commutative ring and $P_1, \ldots, P_n$ be mutually incomparable prime ideals of R. Let $W = S - \bigcup_{i=1}^n P_i$. Prove that the maximal ideals of $R = W^{-1}S$ are precisely the ideals $P_i W^{-1}R$.

(b) Let $S = K[x_1, \ldots, x_d, y]$ be the polynomial ring in $d+1$ variables over a field K. Let $n = 2$, let $P_1 = (x_1, \ldots, x_d)S$ and $P_2 = yS$. Construct $R = W^{-1}S$ as in part (a). What is the Krull dimension of R? What is the Krull dimension R/yR?

2. Is the ring of germs of continuous real-valued functions at the origin in $\mathbb{R}$ a Noetherian ring? Prove your answer.

3. Let $R \subseteq S$ be rings and suppose that there is an R-linear map $\phi : S \to R$ such that $\phi(r) = r$ for all $r \in R$. (We showed in class that this implies $IS \cap R = I$ for all I in R.) We shall say that R is a *direct summand* of S in this situation.

(a) Show that if S is Noetherian then so is R.

(b) Show that $R[[x_1, \ldots, x_n]]$ is a direct summand of $S[[x_1, \ldots, x_n]]$.

(c) Let R_i be an increasing sequence of subrings of the Noetherian ring S such that every R_i is a direct summand of S. Prove that the union $\bigcup_{i=1}^\infty R_i$ is Noetherian.

(d) Let K_i be an increasing sequence of subfields of a field L. Show that the ring $\bigcup_{i=1}^\infty K_i[[x_1, \ldots, x_n]]$ is Noetherian.

4. Let A be a domain with fraction field L and let P be the prime ideal of $B = A[x_1, \ldots, x_n]$ generated by $x_1, \ldots, x_n$, where $x_1, \ldots, x_n$ are variables over A. Show that B_P is isomorphic to $L[x_1, \ldots, x_n]_m$, where $m = (x_1, \ldots, x_n)B$. Hence, B_P is Noetherian. What is its Krull dimension? (A might be $K[z_i : i \in I]$ for an infinite family of new variables over a field K: it need not be Noetherian.)

5. Let R be a ring, and suppose that for every maximal ideal m of R, R_m is Noetherian. Suppose also that every nonzero element of R is in only finitely many maximal ideals. Prove that R is Noetherian.

6. Let K be a field, and let $S = K[x_1, \ldots, x_n, \ldots]$, the polynomial ring in a countably infinite set of variables over K. Partition the variables into sets $V_1, V_2, \ldots, V_n, \ldots$ so that $V_1 = \{x_1\}$ and V_n contains the n variables with smallest indices not in $V_1 \bigcup \cdots \bigcup V_{n-1}$. Thus, $V_2 = \{x_2, x_3\}$, $V_3 = \{x_4, x_5, x_6\}$, and so forth. Let P_n be the prime ideal of S generated by the variables in V_n. Let $W = S - \bigcup_{n=1}^\infty P_n$. Let $R = W^{-1}S$. Show that the maximal ideals of R are precisely the ideals $m_n = P_n S$ (unfortunately, this does not follow from Problem **1.**(a): a separate argument is needed), that R_{m_n} is Noetherian of Krull dimension n, and that every nonzero element of R is in only finitely many maximal ideals of R. Thus R is a Noetherian domain with maximal ideals of arbitrarily large height, and the Krull dimension of R is infinite. Problems **4.** and **5.** are relevant.