Math 614, Fall 2012 **Problem Set #6: Solutions**

1.(a) Since W is the complement of a union of primes it is a multiplicative system, and the primes of $W^{-1}R$ correspond bijectively to the primes of R disjoint from W. This also applies to **6.** These are precisely the primes $\subseteq \bigcup_{i=1}^{n} P_i$ (in **6.**, replace n by ∞). By the theorem on prime avoidance proved in class, these are the same as the prime ideals contained in one of the P_i, from which it is clear that the maximal elements correspond to the P_i themselves (since they are mutually incomparable). $\square$

(b) There are two maximal ideals and by **4.** the localizations at these at P_1 and P_2 have Krull dimensions d and 1, respectively. Hence,Krull dim $R = d$. $R/yR \cong$ the localization of $K[x_1, \ldots, x_d]$ at the image of W, which is $K[x_1, \ldots, x_d] - \{0\}$ (if $f \in K[x_1, \ldots, x_d] - \{0\}$, $f + y \in W$). Thus, R/yR is the field $K(x_1, \ldots, x_d)$, and has Krull dimension 0.

2. This ring, call it S, is not Noetherian: the ideals $[f_n]S$ ([ ] indicates"germ of") form a strictly ascending infinite chain, where for all $x \in \mathbb{R}$, $f_n(x) = |x|^{1/2^n}$, so that $f_n = f_{n+1}^2$ for all n. The inclusions are strict, for if $f_{n+1} = gf_n$ then $g(x) = 1/f_{n+1}(x)$ when $x \neq 0$, and $g(x) \to +\infty$ as $x \to 0$.

3. (a) Given an ascending chain I_n in R, the I_nS are ascending in S, and so $I_nS = I_{n+k}S$ for some n and all $k \geq 0$. Hence $I_n = I_nS \cap R = I_{n+k}S \cap R = I_{n+k}$ for the same value of n and all $k \geq 0$. $\square$

(b) If $\theta : S \to R$ is an R-linear map that fixes every element of R, then $\Theta : S[[x]] \to R[[x]]$ defined by $\Theta(\sum_{n=0}^{\infty} r_n x^n) = \sum_{n=0}^{\infty} \theta(r_n)x^n$ is easily checked to be an $R[[x]]$-linear map of $S[[x]]$ to $R[[x]]$, and clearly fixes every element of $R[[x]]$. $\square$

(c) Let I be an ideal of the union $R = \bigcup_{i=1}^{\infty} R_i$. Since S has ACC, IS is generated by a finitely many elements $f_1, \ldots, f_s$ of I, and we can choose $k \gg 0$ such that $f_1, \ldots, f_s \in R_k$. Let $J = (f_1, \ldots, f_s)R$. Clearly, $J \subseteq I$. We claim $J = I$. If $r \in I$, choose $n \geq k$ with $r \in R_n$. Then $r \in IS \cap R_n = (f_1, \ldots, f_s)S \cap R_n = (f_1, \ldots, f_s)R_n \subseteq (f_1, \ldots, f_s)R = J$. $\square$

(d) Since K_i is a direct summand of L for all i (L is a K_i-vector space), (b) implies that $K_i[[x]]$ is a direct summand of $L[[x]]$ for all i, and the result is immediate from part (c). $\square$

4. The elements of $A - \{0\}$ are in $R - P$ and so $L \subseteq R_P$. Hence, $L[x_1, \ldots, x_n] \subseteq R_P$. The image of $R - P$ in $S = L[x_1, \ldots, x_n]$ is in $S - Q$, where $Q = (x_1, \ldots, x_n)S$, and every element of $S - Q$ can be multiplied into $R - P$ by an element of $A - \{0\}$. Hence, $R_P = (R - P)^{-1}R = (S - Q)^{-1}S = S_Q$, as required. Since Q has height n in S, $S_Q \cong R_P$ has Krull dimension n. $\square$

5. We show every ideal $I \subseteq R$ is finitely generated. We may assume that $I \neq (0)$. Let $f \in I - \{0\}$. Let $m_1, \ldots, m_n$ be the maximal ideals of R with $f \in m_i$. For each j, $1 \leq j \leq n$, let $f_{i,1}/w_{i,1}, \ldots, f_{i,h_i}/w_{i,h_i}$ be a finite set of generators of IR_{m_i}, where $f_{ij} \in I$ and $w_{ij} \in R - m_i$. Since the $w_{i,j}$ are units in R_{m_i}, the images of the f_{ij} generate IR_{m_i}. Let $J \subseteq R$ be the ideal generated the $(\sum_{i=1}^{n} h_i) + 1$ elements f_{ij} together with f. Clearly, $J \subseteq I$. We show that $J = I$, i.e., that $I/J = 0$. If not, $(I/J)_m = I_m/J_m \neq 0$ for some maximal ideal $m \subseteq R$. But if $f \notin m$, then $JR_m = IR_m = R_m$, while if $f \in m$ then $m = m_i$ and $J_m = I_m$ because the $f_{ij} \in J$. $\square$

Note: It is *not* true that if $I_1 \subseteq \cdots \subseteq I_n \subseteq \cdots$ are contained in a maximal ideal m and the expansions stabilize in R_m, then the chain stabilizes in R: there were some erroneous proofs trying to use this. Ideals of mR_m correspond bijectively to ideals of R contained in m and *contracted* with respect to $R - m$, not to all ideals contained in m. Let K be a field and let R consist of K-valued functions on $\mathbb{N}$ that are eventually constant, and let m consist of functions that are eventually 0 ($R/m \cong K$). Let I_n be all functions that vanish for integers $\geq n$. All $In \subseteq m$ and $\{I_n\}_n$ is strictly ascending, while for all n, $I_n R_m = (0)$.

6. We first need to show that the maximal ideals of R are the $P_n R$. This follows as in **1.**(a) if we know that whenever $m \subseteq \bigcup_{n=1}^{\infty} P_n$, then $m \subseteq P_n$ for some n. Let f be a nonzero element of m and pick h so that all variables that occur in f are among the generators of $P_1, \ldots, P_h$. Let $Q = \sum_{n=h+1}^{\infty} P_n$. Then $m \subseteq Q \bigcup \left(\bigcup_{n=1}^{h} P_n\right)$. Clearly, $f \notin Q$, so by prime avoidance, $m \subseteq P_i$, $1 \leq i \leq h$. It then follows from **4.** that $R_{P_n R} \cong S_{P_n}$ is Noetherian of Krull dimension n for every n. It remains only to check that R is Noetherian, which will follow from **5.** provided that every element $f/w \neq 0$, where $f \in S - \{0\}$, $w \in W$, is in only finitely many primes of R. It suffices to check that for every $f \in S - \{0\}$, f is in only finitely many of the P_n. This was shown above. $\square$