

Due: Monday, September 23

1. Let R be a commutative ring and suppose that there is a surjective ring homomorphism of R onto an integral domain D that is not a field. Prove that $\text{Spec}(R)$ is not a Hausdorff space.

2. Let $\mathbb{C}[x]$ be the polynomial ring in 1 variable over \mathbb{C} , and for $n \geq 1$, an integer, let A_n be the subring of $\mathbb{C}[x]$ generated by \mathbb{C} and the polynomials $x^{k+1} - x^k$, $1 \leq k \leq n$. E.g., $A_1 = \mathbb{C}[x^2 - x] \subseteq \mathbb{C}[x]$ and $A_2 = \mathbb{C}[x^3 - x^2, x^2 - x] \subseteq \mathbb{C}[x]$, and so on. Which, if any, of the rings A_n are equal to $\mathbb{C}[x]$? Determine for every n a basis for the \mathbb{C} -vector space $\mathbb{C}[x]/A_n$.

In problems **3.** and **4.**, let $S = K[x_0, \dots, x_n, \dots]$ be the polynomial ring in a countably infinite sequence of variables over a field K .

3. Let $c_n \in K - \{0\}$ for every $n \in \mathbb{N}$ and let I be the ideal of S generated by all the polynomials $x_n^2 - c_n x_n$, $n \in \mathbb{N}$. Let $R = S/I$. Give an explicit description of the prime ideals of R . Which of these are maximal?

Extra Credit 1. Show that $\text{Spec}(R)$ in Problem **3.** is homeomorphic with the product of countably many discrete topological spaces each of which has two points. (This space is in turn homeomorphic with the Cantor set in the unit interval.)

4. Let J be the ideal of S generated by all the polynomials $x_m x_n$, $m, n \in \mathbb{N}$. Let $R = S/J$. Give a complete description of $\text{Spec}(R)$. Is every ideal of R finitely generated? (Prove your answer.)

5. Let S be and \mathfrak{A} be subsets of the ring R . Suppose that for every choice of finite sets S_0, \mathfrak{A}_0 with $S_0 \subseteq S$ and $\mathfrak{A}_0 \subseteq \mathfrak{A}$, there exists a prime ideal of R that contains \mathfrak{A}_0 but is disjoint from S_0 . Prove that there exists a prime ideal of R that contains \mathfrak{A} but is disjoint from S .

6. Let K be an algebraically closed field and let $R = K[x, y]$ be a polynomial ring in two variables over K . Consider the subring S of R generated by K , $u = xy - 1$, and $v = y + x(xy - 1)$, i.e., $S = K[u, v] = K[xy - 1, y + x(xy - 1)]$. You may assume that u and v are algebraically independent, so that S is also a polynomial ring in two variables. For which $(a, b) \in K^2$ is $(u - a, v - b)$ the contraction of a maximal ideal of S of the form $(x - c, y - d)$, where $c, d \in K$? (Later we'll prove that all maximal ideals of R and of S have the specified forms.)

Extra Credit 2. Give, if possible, an example of an integral domain R in which $\text{Spec}(R)$ consists of the prime ideals in one strictly ascending infinite sequence

$$0 = P_0 \subset P_1 \subset P_2 \subset \dots \subset P_n \subset \dots$$

together with one maximal ideal m such that $m = \bigcup_{n=0}^{\infty} P_n$. Describe the Zariski topology on $\text{Spec}(R)$.