Math 614, Fall 2013 **Problem Set #2**

Due: Wednesday, October 9

1. If R is a ring, $f \in R$ and $I \subseteq R$, $I :_R f$ denotes the ideal $\{r \in R : rf \in I\}$.

(a) Prove that for each prime P of R, the image of f in R_P is *not in* IR_P iff $P \supseteq I :_R f$.

(b) Let K be a field, let $R = K[x_1, \ldots, x_n]$ be a polynomial ring, let $I = (x_1^2, \ldots, x_n^2)$ and let $f = x_1 + \cdots + x_n$. Determine generators for $I :_R f$ for $n \leq 4$. Additional credit will be given for analysis for larger n. (The answer may depend on char(K).)

2. Let R be a nonzero reduced commutative ring with only finitely many prime ideals, all of which are maximal. Show that R is isomorphic with a finite product of fields.

3. Let R be a nonzero reduced ring with only finitely many minimal primes. Let W be the multiplicative system consisting of all elements not in any minimal prime. Show that every element of W is a nonzerodivisor in R. (Hence, R injects into $W^{-1}R$.) Prove that $W^{-1}R$ is a finite product of fields.

4. (a) Let R be a ring, $W \subseteq R$ a multiplicative system, and $S = W^{-1}R$. Let $f : M \to N$ be an R-linear map of S-modules. Show that f is S-linear, i.e., $\mathrm{Hom}_R(M, N) = \mathrm{Hom}_S(M, N)$.

(b) Let R be the polynomial ring $K[x, y]$ over a field K and S be $K[x, y/x]$ (a subring of the fraction field of R). Let $v = y/x \in S$. Note that $K[x, v]$ is also a polynomial ring in two variables. Let $M = S/xS$. Is $\mathrm{Hom}_R(M, S) = \mathrm{Hom}_S(M, S)$? Prove your answer. [Later EC: Is $\mathrm{Hom}_R(S, M) = \mathrm{Hom}_S(S, M)$: in any case, describe both.]

5. If P is a prime ideal of R, $P^{(n)}$ denotes the contraction of $P^n R_P$ to R, and is called the n th *symbolic power* of P. Let $T = K[u, v, w, x, y, z]$ be a polynomial ring over a field K, and let $f = ux + vy + wz$. Let $R = T/fT$. Let P be the ideal of R generated by v, w, x, y, and z. Show that P is prime, and that $P^{(2)} \neq P^2$.

6. Let R be a ring and $W \subseteq R$ a multiplicative system. Let $S = W^{-1}R$. Let M and N be R-modules. Note that there is an S-linear map $\theta : W^{-1}\mathrm{Hom}_R(M, N) \to \mathrm{Hom}_S(W^{-1}M, W^{-1}N)$ such that $[f/w] \mapsto (1/w)W^{-1}f$, where $W^{-1}f$ is as described in class. Show that if $R = K[x_1, \ldots, x_n, \ldots]$ is the polynomial ring in a countably infinite sequence of variables over a field K, W is the set of powers of x_1, $M = R/I$, where $I = (x_n : n \geq 2)R$, and $N = R/J$, where $J = (x_1^n x_n : n \geq 2)R$, and then the map the map θ is not onto: in fact, show that there is an isomorphism $W^{-1}M \cong W^{-1}N$ that is not in the image of θ. (Later, we'll give a condition that is sufficient for θ to be an isomorphism.)

Extra Credit 3. Let M be a module over a ring R. Suppose that M_P is generated as an R_P-module by at most one element for every prime P of R. Must M be a finitely generated R-module? Prove your answer.

Extra Credit 4. An integral domain R is said to be *normal* if it contains every element f of its fraction field that satisfies a monic polynomial with coefficients in R. Suppose that R is a domain that satisfies the weaker condition that whenever f is in the fraction field and $f^n \in R$ for some $n \in \mathbb{Z}_+$ then $f \in R$. Must R be normal? Prove your answer.