Math 614, Fall 2013 **Problem Set #4**
Due: Friday, November 22

1. Let x and y be relatively prime elements in a UFD R such that $I = (x, y)R$ is a proper ideal. You may assume that the surjection $R^2 \twoheadrightarrow I$ that sends $(r, s) \mapsto sx - ry$ has kernel Rv spanned by $v = (x, y) \in R^2$, so that $I \cong R^2/Rv$. Represent $I \otimes_R I$ as the cokernel of a map of finitely generated free modules. Prove that the map $I \otimes I \twoheadrightarrow I^2$ has a nonzero kernel spanned by $\theta = x \otimes y - y \otimes x \in I \otimes_R I$, and that the annihilator of θ in R is I.

2. Let $X = (x_{ij})$ be an $n \times n$ matrix of indeterminates over an algebraically closed field K. Show that the algebraic set Z_n in $\mathbb{A}_K^{n^2}$ defined by the vanishing of the entries of the matrix X^n can also be defined by the vanishing of n polynomials in the x_{ij}. Show that Z_n is irreducible, i.e., is a variety. (For the latter, it may be useful to show that Z_n is the image of $\mathrm{GL}(n, K) \times Y$ under a regular map, where Y consists of the upper triangular matrices with 0 entries on the main diagonal.) What is the dimension of the variety Z_n?

3. Let M be a finitely generated module over a Noetherian ring R. Show that the set $\{P \in \mathrm{Spec}\,(R) : M_P \text{ is free over } R_P\}$ is Zariski open in $\mathrm{Spec}\,(R)$.

4. Let $K_1 \subset \cdots \subset K_n \subset \cdots$ be an infinite chain of proper field extensions. Let x be a power series indeterminate over all the K_n. Let $R = \bigcup_{n=1}^{\infty} K_n[[x]]$. Show that R is a local (Noetherian) domain in which every nonzero proper ideal is generated by a power of x.

5. Let f and g be elements of a ring R such that $f + g = 1$. Let M be an R-module. Suppose that $u, v \in M$ are such that $gu - fv = 0$. Show that there is a unique element $m \in M$ such that the image of m in M_f is u/f and the image of m in M_g is v/g.

6. Let R be any ring, and let $Y \subseteq X =: \mathrm{Spec}\,(R)$ be a subset that is a finite union of sets of the form $U_i \cap Z_i$, where U_i is quasicompact and open in X and Z_i is closed in X. Show that there is a finitely generated R-algebra S such that the image of the map $\mathrm{Spec}\,(S) \to \mathrm{Spec}\,(R)$ is Y. [Suggestion: if $R \to S_1$ and $R \to S_2$ are homomorphisms, compare the image of $\mathrm{Spec}\,(S_1 \times S_2)$ with the images of $\mathrm{Spec}\,(S_1)$ and $\mathrm{Spec}\,(S_2)$.]

Extra Credit 8. Let L be a field extension of K that is not finitely generated as a field extension. Prove that $L \otimes_K L$ is not a Noetherian ring.

Extra Credit 9. Let $K_1 \subset \cdots \subset K_n \subset \cdots$ be as in **3.** and let $x_1, \ldots, x_d$ be power series indeterminates over all the of the fields K_n. Let $R_d = \bigcup_{n=1}^{\infty} K_n[[x_1, \ldots, x_d]]$. Is R_d Noetherian? Prove your answer.