Math 614, Fall 2013 **Problem Set #5**

Due: Wednesday, December 11

1. Let $R = K[w, x, y, z]$, a polynomial ring. Find an irredundant primary decomposition for $(x^3, xy^5z, x^2yw, z^7w^7)R$ consisting of ideals generated by monomials. Which ideals are unique in your primary decomposition?

2. Let f be an element of a polynomial ring S over R with a coefficient that is a unit in R. Prove that if $g \in S$ and $fg = 0$, then $g = 0$.

3. Let R be a K-algebra, where K is a field, and let L be a field extension of R. Suppose that $T = L \otimes_K R$ is normal. Prove that R is normal. Prove also that if T is a Dedekind domain, then R is a Dedekind domain.

4. Let $\mathbb{R}$ be the real numbers, let $S = \mathbb{R}[X, Y]/(X^2 + Y^2 - 1) = \mathbb{R}[x, y]$, where X and Y are indeterminates. Let $T = \mathbb{C} \otimes_{\mathbb{R}} S \cong \mathbb{C}[X, Y]/(X^2 + Y^2 - 1)$.

(a) Prove that $T \cong \mathbb{C}[u, 1/u]$ where $u = x + yi$, and so is a PID. Prove that R is a Dedekind domain.

(b) Let $m = (x - 1, y)S$, which is a maximal ideal of S. Show that mT is principal and exhibit a generator, $f \in T$. Prove that m is not principal by showing that f has no unit multiple in S. Prove that m^2 is principal.

(c) Prove that $m \oplus m \cong S \oplus S$.

5. Let R be any commutative ring. Suppose that R has Krull dimension n. Let $S = R[x]$ be the polynomial ring in one variable over R. Show that $n+1 \le \dim(S) \le 2n+1$. (This is known to be the sharpest statement that can be made.)

6. Let R be a ring in which every prime ideal is finitely generated. Prove that R is Noetherian. (Suggestion: use Zorn's lemma to show that if R is not Noetherian there exists a proper ideal I that is not finitely generated but such that every strictly larger ideal is finitely generated. Get a contradiction by proving that I is prime. Note if $f, g \notin I$ but $fg \in I$, one may consider the ideals $I + fR$ and $I :_R f$.)

Extra Credit 10. Let R a ring, let K be a field, let z be an indeterminate over R and let x, y be indeterminates over K. Suppose that $R[z] \cong K[x, y]$. Prove that R is a polynomial ring in one variable over K.

Extra Credit 11. Let $0 \to A \to B \to C \to 0$ be an exact sequence of modules of finite length over a ring R. Suppose that $B \cong A \oplus C$ (but the decomposition is not necessarily related to the given exact sequence). Show that the given exact sequence splits.

Extra Credit 12. Let K be a field of characteristic not 2. Let $R = K[x, y]/fK[x, y]$, where x and y are indeterminates and $f = y^2 - x^2 - x^3$. Show that R is a domain, but that its completion with respect to the maximal ideal m generated by the images of x and y is not.